STRATEGY-PROOF RANDOM VOTING RULES ON WEAK DOMAINS

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Abstract

We consider domains with weak preferences and provide the structure of unanimous and strategy-proof random social choice functions on these domains. We begin with weak single-peaked domains and provide a characterization of Pareto optimal and strategy-proof random functions on these domains. Next, we consider single-plateaued domains. First, we show that every unanimous and strategy-proof random social choice function on such domains satisfies Pareto optimality and almost plateau-onlyness, and then provide an axiomatic characterization of these functions. We also provide a functional form characterization of anonymous, plateau-only, and strategy-proof random social choice functions on the single-plateaued domains under some restrictions on the plateau size.

JEL Classification: D78; D81

Keywords: Strategy-proofness; Single-peaked preferences; Single-plateaued preferences; Pareto optimality; Random social choice functions

1. INTRODUCTION

We consider the standard social choice problem where a social planner has to choose an alternative from a feasible set based on the preferences of the agents in a society. Unanimity, Pareto optimality, strategy-proofness are considered as desirable properties of a deterministic social choice function (DSCF). A DSCF

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is unanimous if, whenever all agents agree on their top-ranked alternative, that alternative is chosen. It is Pareto efficient if its outcome cannot be improved in way so that no one is worse off and someone is better off. It is strategy-proof if no agent can benefit by misreporting her preference.

The horizon of social choice theory is expanded by introducing randomness in social choice functions. A random social choice function (RSCF) selects a probability distribution over the alternatives at every collection of preferences. The notions of unanimity and Pareto optimality remain the same for RSCFs, while that of strategy-proofness is formulated by means of stochastic dominance.

Importance of RSCFs over DSCFs is well-established in the literature (see Peters et al. (2017), Peters et al. (2014) and Ehlers et al. (2002)). The appeal of RSCFs over DSCFs is that they allow for the introduction of fairness considerations and reasonable compromise in the decision-making process, facilitating the resolution of conflicts of interest (see Nandeibam (2013) and Picot and Sen (2012)). Apart from providing fair outcomes (see Ehlers et al. (2002) for details), they also increase the ex-ante welfare of a society over DSCFs.

A preference is strict if it does not admit any indifference. The study of social choice functions when preferences are strict is quite extensive. In a seminal work, Gibbard et al. (1977) shows that an RSCF on the strict unrestricted domain is unanimous and strategy-proof if and only if it is random dictatorial. Subsequently, several other random dictatorial domains are characterized in the literature. For domains with specific structure, unanimous and strategy-proof RSCFs on the strict single-peaked domain, the strict single-dipped domain, and strict single-crossing domains are characterized (see Ehlers et al. (2002), Peters et al. (2017), Roy and Sadhukhan (2021)).

To our understanding, the assumption of strict preferences is quite strong for practical purposes. It is a well-established fact that individuals' preferences admit indifferences. However, contrary to their strict counterpart, the literature of social choice theory for weak preferences is rather limited. Our objective of this paper is to explore this area of social choice for weak preferences.

Next, we analyze what happens when weak preferences are added to well-known restricted domains. We consider two types of weak domains in this context: one where indifference occurs only at the top-position, and the other one where it occurs only below the top-position. An important weak domain of the former type is the single-plateaued domain and that of the latter type is the single-peaked domain with outside option. We provide the structure of unanimous (or Pareto optimal) and strategy-proof RSCFs for each of these cases. Berga (1998) provide the structure of unanimous and strategy-proof DSCFs on the single-plateaued domain and Cantala (2004) provide that for the single-peaked domain with outside option. We generalize these results for RSCFs. Also, we provide closed form presentation of these rules.

In Section 2 we introduce the basic model, notations and definitions. In Section 3 we characterize the Pareto efficient and strategy-proof RSCFs as extreme PFBRs on weak single-peaked domains. In Section 4 we consider single-plateaued domains and show that every unanimous and strategy-proof random social choice function on such domains satisfies Pareto optimality and almost plateau-onlyness, and then provide an axiomatic characterization of these functions. We also provide a functional form characterization of anonymous, plateau-only, and strategy-proof random social choice functions on the single-plateaued domains under some restrictions on the plateau size. In Section 5 we provide the conclusion.

2. PRELIMINARIES

Let $N = \{1,\ldots,n\}$ be a set of at least two agents, who collectively choose an element from a finite set $A = \{1,\ldots,m\}$ of at least two alternatives. For $x,y \in A$ such that $x \leq y$, we define the intervals $[x,y] = \{z \in A \mid x \leq z \leq y\}$, $[x,y) = [x,y] \setminus \{y\}$, $(x,y] = [x,y] \setminus \{x\}$, and $(x,y) = [x,y] \setminus \{x,y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i. A (weak) *preference* R over A is a complete and transitive binary relation (also called a weak order) defined on A. We denote by P and I the anti-symmetric and the indifference part of R, respectively. That is, xPy means xRy and not yRx, and xIy means xRy and yRx. We denote by $\mathbb{W}(A)$ the set of all preferences over A. For a preference $R \in \mathbb{W}(A)$, we denote by $\tau(R)$ the set of alternatives that appear at the top-position of R, that is, $\tau(R) = \{x \in A \mid xRy \text{ for all } y \in A\}$. For a preference $R \in \mathbb{W}(A)$ and an alternative x, the upper contour set $U(x,R)$ of R at x is defined as the set of alternatives that are (weakly) preferred to x at R, that is, $U(x,R) = \{y \in A \mid yRx\}$. An antisymmetric preference is called *strict preference*. We denote a strict preference by P.

We denote a set of admissible preferences of an agent i by $\mathscr{D}_i$, and a set of admissible strict preferences by $\widehat{\mathscr{D}}_i$. Let $\mathscr{D}_N = \prod_{i \in N} \mathscr{D}_i$. An element of $\mathscr{D}_N$ is called a preference profile. For ease of presentation, we refer to sets $\mathscr{D}_i$ and $\mathscr{D}_N$ as domains, and sets $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_N$ as strict domains. Furthermore, for a domain $\mathscr{D}_i$ or $\mathscr{D}_N$, we denote by strict$(\mathscr{D}_i)$ and strict$(\mathscr{D}_N)$ the set of strict preferences in $\mathscr{D}_i$ and $\mathscr{D}_N$, respectively.

A random social choice function assigns a probability distribution over the alternatives at each preference profile.

Definition 2.1. A *random social choice function* (RSCF) φ on $\mathscr{D}_N$ is a mapping $\varphi : \mathscr{D}_N \to \Delta A$.

Unanimity says that whenever all the agents in a society have some alternative(s) common in their top position, those alternatives are chosen with probability 1.

Definition 2.2. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is *unanimous* if for all $R_N \in \mathscr{D}_N$ such that $\cap_{i\in N}\tau(R_i) \neq \emptyset$, we have $\sum_{x\in\cap_{i\in N}\tau(R_i)} \varphi_x(R_N) = 1$.

We say an alternative x Pareto dominates another alternative y at a preference profile R_N if every agent weakly prefers x to y and some agent strictly prefers x to y, that is, xR_iy for all $i \in N$ and xP_iy for some $i \in N$. An alternative is said to be Pareto dominated if some other alternative Pareto dominates it. Pareto optimality says that a Pareto dominated alternative cannot be selected with positive probability.

Definition 2.3. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ satisfies *Pareto optimality* if for all $R_N \in \mathscr{D}_N$, we have $\varphi_x(D_N) = 0$ for all $x \in A$ such that x is Pareto dominated at R_N.

An RSCF is strategy-proof if no agent can increase the probability of any upper contour set by misreporting his/her preferences.

Definition 2.4. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called *strategy-proof* if for all $i \in N$, $(R_i, R_{N\setminus i}) \in \mathscr{D}_N$, $R'_i \in \mathscr{D}_i$, and $x \in A$

$$\sum_{y\in U(x,R_i)} \varphi_y(R_i, R_{N\setminus i}) \geq \sum_{y\in U(x,R_i)} \varphi_y(R'_i, R_{N\setminus i}).$$

REMARK 2.1. An RSCF is called a deterministic social choice function (DSCF) if it selects a degenerate probability distribution at every preference profile. More formally, an RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called a DSCF if $\varphi_a(R_N) \in \{0,1\}$ for all $a \in A$ and all $R_N \in \mathscr{D}_N$. The notions of unanimity, Pareto optimality, and strategy-proofness for DSCFs are special cases of the corresponding definitions for RSCFs.

3. RANDOM RULES ON SINGLE-PEAKED DOMAINS

In this section, we consider single-peaked domains and provide a characterization of Pareto optimal and strategy-proof RSCFs on these domains.

A preference is called single-peaked if there is a unique top-ranked alternative such that preference *weakly* declines as one moves away from the top-ranked alternative in any direction.

Definition 3.1. A preference R is called single-peaked if it has a unique top-ranked alternative $\tau(R)$, called the peak, such that for all $a, b \in A$, $b < a < \tau(R)$ or $\tau(R) < a < b$ implies aRb. A domain is called single-peaked if each preference in it is single-peaked.

A single-peaked preference is called strict single-peaked if it does not contain any indifference. A domain is called strict single-peaked if each preference in it is strict single-peaked.

Next, we introduce the notion of probabilistic fixed ballot rules. We need the following terminology. For a preference profile R_N and an alternative x, we denote the set of agents whose peaks are on the right of x by $S(x,R_N)$, that is, $S(x,R_N) = \{i \in N \mid \tau(R_i) \geq x\}$.

Definition 3.2. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called a probabilistic fixed ballot rule if for all $S \subseteq N$, there exists $\beta_S \in \Delta A$ satisfying

(i) $\beta_N(m) = 1$ and $\beta_\emptyset(1) = 1$, and

(ii) $\beta_S([x,m]) \leq \beta_T([x,m])$ for all $S \subseteq T$ and all $x \in A$

such that for all $R_N \in \mathscr{D}_N$ and all $x \in A$, we have

$$\varphi_x(R_N) = \beta_{S(x,R_N)}[x,m] - \beta_{S(x+1,R_N)}[x+1,m],$$

where $\beta_{S(m+1,R_N)}[m+1,m] \equiv 0$.

Finally, we introduce the notion of extreme PFBRs. These are special cases of PFBRs where each β_S assigns positive probabilities only to the "extreme" alternatives 1 and m.

Definition 3.3. A PFBR with respect to parameters $(\beta_S)_{S\subseteq N}$ is called extreme if $\beta_S(x) > 0$ implies $x \in \{1,m\}$.

A single-peaked preference R is called left dichotomous if $\tau(R) = 1$ and $2Im$. In other words, except from alternative 1, which is ranked top at R, all other alternatives are indifferent to each other. Similarly, a preference R' is called right dichotomous if $\tau(R') = m$ and $(m-1)I'1$. A single-peaked domain is minimally rich if it contains all strict single-peaked preferences, the left dichotomous, and the right dichotomous preference. Our next theorem says that every Pareto optimal and strategy-proof RSCF on the single-peaked domain is an extreme PFBR. Cantala (2004) considers the single-peaked domain with outside options and characterizes Pareto optimal and strategy-proof DSCFs on those domain.[1] The single-peaked domains with outside option are special cases of minimally rich weak single-peaked domains, and hence a characterization of Pareto optimal and strategy-proof RSCFs on those domains follows as a corollary of our result.

Theorem 3.1. *Let $\mathscr{D}_i$ be a minimally rich single-peaked domain for each $i \in N$. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is Pareto optimal and strategy-proof if and only if it is an extreme PFBR.*

The proof of the theorem is relegated to Appendix A.

[1] A preference is single-peaked with outside options if there is a region around the peak such that the preference exhibits single-peakedness over the alternatives in that region and indifference over the ones outside it.

4. Random rules on single-plateaued domains

In this section, we introduce the notion of single-plateaued preferences. For these preferences, an interval of alternatives appear at the top-position, and as one go far from this interval (in any particular direction), preference declines *strictly*. Moreover, indifference occurs only at the top-position for such a preference. Note that single-plateaued preferences are the counter part of weak single-peaked preferences in the sense that for the former, indifference can occur only at the top, whereas for the latter, it can occur only below the top.

Throughout this section, we assume that admissible preferences are the same across agents.

Definition 4.1. A preference $R \in \mathbb{W}(A)$ is called *single-plateaued* if there exist $x, y \in A$ with $x < y$ such that

(i) $\tau(R) = [x, y]$,

(ii) for all $u, v \in A$, $[u < v \leq x$ or $y \leq v < u]$ implies vPu, and

(iii) for all $u, v \notin [x, y]$, either uPv or vPu.

In what follows, we introduce some particular type of single-plateaued preferences based on the size of the plateau.

Definition 4.2. For $1 \leq \kappa_1 \leq \kappa_2 < m$, a single-plateaued preference $R \in \mathbb{W}(A)$ is called (κ_1, κ_2)-*single-plateaued* if $\kappa_1 \leq |\tau(R)| \leq \kappa_2$. A domain is called (κ_1, κ_2)-single-plateaued domain if it contains all (κ_1, κ_2)-single-plateaued preferences.

For a single-plateaued preference R, we denote by $\tau^+(R)$ and $\tau^-(R)$ the right-end point and the left end-point of the plateau, respectively. More formally, if $\tau(R) = [x, y]$, then $\tau^+(R) = y$ and $\tau^-(R) = x$.

4.1 Equivalence of unanimity and Pareto optimality under strategy-proofness

In this section, we introduce the concepts of unanimity and Pareto optimality. Unanimity is a weaker notion of Pareto optimality, however we show that under strategy-proofness they are equivalent on a single-plateaued domain. It is worth mentioning that the same result holds on a single-peaked domain (see Moulin (1980) and Weymark (2011)).

From this section onward, we assume that all the agents have the same set of admissible preferences.

Our next theorem says that unanimity and Pareto optimality are equivalent for a strategy-proof RSCF on a (κ_1, κ_2)-single-plateaued domain.

Theorem 4.1. *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathscr{D}$ be a (κ_1, κ_2)-single-plateaued domain. Suppose $\varphi : \mathscr{D}^n \to \Delta A$ is a strategy-proof RSCF. Then, φ is unanimous if and only if it is Pareto optimal.*

The proof of the theorem is relegated to Appendix B.

4.2 Unanimity and almost plateau-onlyness

In this section, we analyze the connection between unanimity and a weaker version of plateau-onlyness called almost plateau-onlyness in the presence of strategy-proofness. An RSCF is called plateau-only if its outcome depends only on the plateaus at a preference profile.

Definition 4.3. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called *plateau-only* if for any two preference profiles $R_N, R'_N \in \mathscr{D}^n$, $\tau(R_i) = \tau(R'_i)$ for all $i \in N$ implies $\varphi(R_N) = \varphi(R'_N)$.

On single-peaked domains, peaks-onlyness and unanimity are equivalent for random rules under strategy-proofness (Moulin (1980) and Weymark (2011)). However, as the following example suggests, the same does not hold even for deterministic rules on single-plateaued domains when peaks-onlyness is replaced by plateau-onlyness.

Example 4.1. Consider the DSCF, say f, given in Table 1. It can be verified that φ is unanimous and strategy-proof. However, since $\tau([23]14) = \tau([23]41)$ and $f([23]14, [123]4) \neq f([23]41, [123]4)$, it is not plateau-only. □

It is worth noting from Example 4.1 that if an agent changes his/her preference maintaining his/her plateau, then unanimity and strategy-proofness can never rule out the possibility of rearranging the probabilities of the alternatives in the plateau. In view of this fact, we weaken the notion of plateau-onlyness by almost plateau-onlyness. It says that if an agent changes his/her preference maintaining his/her plateau, then the probability of any alternative that lies *outside* his/her plateau must remain the same. In other words, the only change that can happen by this change of preference is that the probabilities of the alternatives in his/her plateau are rearranged.

Definition 4.4. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called *almost plateau-only* if for any two preference profiles $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in \mathscr{D}^n$, $\tau(R_i) = \tau(R'_i)$ implies $\varphi_x(R_i, R_{N\setminus i}) = \varphi_x(R'_i, R_{N\setminus i})$ for all $x \notin \tau(R_i)$.

1 \ 2	1234	2134	2314	3214	3421	4321	[12]34	[23]14	[23]41	[34]21	[123]4	[234]1
1234	1	2	2	2	2	2	1	2	2	2	1	2
2134	2	2	2	2	2	2	2	2	2	2	2	2
2314	2	2	2	2	2	2	2	2	2	2	2	2
3214	2	2	2	3	3	3	3	3	3	3	3	3
3421	2	2	2	3	3	3	3	3	3	3	3	3
4321	2	2	2	3	3	4	2	3	3	4	3	4
[12]34	1	2	2	3	3	2	1	2	2	2	2	2
[23]14	2	2	2	3	3	3	2	3	2	3	3	2
[23]41	2	2	2	3	3	3	2	2	3	3	2	3
[34]21	2	2	2	3	3	4	2	3	3	4	3	3
[123]4	1	2	2	3	3	3	1	3	2	3	1	3
[234]1	2	2	2	3	3	4	2	3	2	4	2	3

Table 1

Theorem 4.2. *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathscr{D}$ be a (κ_1, κ_2)-single-plateaued domain. Let $\mathscr{D}$ be a (κ_1, κ_2)-single-plateaued domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Then, φ is almost plateau-only.*

The proof of the theorem is relegated to Appendix C.

4.3 A CHARACTERIZATION OF UNANIMOUS AND STRATEGY-PROOF RULES ON ARBITRARY SINGLE-PLATEAUED DOMAINS

In this section, we provide a characterization of unanimous and strategy-proof RSCFs on arbitrary single-plateaued domains. We do this by identifying a property called generalized uncompromisingness of such rules. As the name suggests, this property is a generalization of the uncompromisingness property that exists in the literature in the context of single-peaked domains.

The notion of generalized uncompromisingness turns out to be relatively simpler for DSCFs. To help the reader, we first present this notion for DSCFs.

A DSCF satisfies generalized uncompromisingness if the following happens. Whenever an agent unilaterally moves his/her plateau in some direction, (i) if both the plateaus lie either strictly on the right of the outcome or strictly on the left of the outcome of the DSCF, then the outcome does not change, and (ii) if the right-end point or the left-end point of the plateau crosses the outcome, then the outcome moves in the direction to which the plateau has moved.

Definition 4.5. An DSCF $f : \mathscr{D}^n \to A$ satisfies generalized uncompromisingness if for all $R_i, R'_i \in \mathscr{D}$, and all $R_{N \setminus i} \in \mathscr{D}^{n-1}$, we have

(i) $[f(R_N), f(R'_N) \leq \min\{\tau^-(R_i), \tau^-(R'_i)\}]$ or $[\max\{\tau^+(R_i), \tau^+(R'_i)\} \leq f(R_N), f(R'_N)]$, and

(ii) $[\tau^+(R_i) < f(R_N) \leq \tau^+(R_i')]$ or $[\tau^-(R_i) < f(R_N) \leq \tau^-(R_i')]$ implies $f(R_N) \leq f(R_i', R_{N\setminus i})$.

We illustrate the notion of generalized uncompromisingness by means of the following example. It should be noted that DSCFs satisfying generalized uncompromisingness can be constructed in a relatively easy manner.

Example 4.2. Let the set of alternatives be $A = \{1,2,3,4,5\}$ and suppose that there are two agents $N = \{1,2\}$. We consider an arbitrary single-plateaued domain $\mathscr{D}$. In Table 2, we present a DSCF, say f, that satisfies generalized uncompromisingness. To see that f satisfies part (i) of generalized uncompromisingness, consider, for instance, the preference profiles $(12345, [23]145)$ and $(12345, [45]321)$. Note that agent 2 changes his/her plateau from $[23]$ to $[45]$ from the former preference profile to the latter. The outcome at the former preference profile is 1, which lies strictly on the left of both the plateaus $[23]$ and $[45]$. As required by (i), the outcome at the latter preference profile is also 1. To see that f satisfies part (ii) of generalized uncompromisingness, consider, for instance, the preference profiles $([123]45, [123]45)$ and $([123]45, [45]321)$. Note that the outcome at the former preference profile is 2, which lies (weakly) on the right of the former plateau 1 and strictly on the left of the latter plateau 4. As required by (ii), the outcome moves to its right from 2 to 3. It is worth mentioning that although the DSCF in this example is chosen to be unanimous, unanimity is not implied by generalized uncompromisingness. Later, we will make a formal remark to emphasize this fact. □

1 \ 2	12345	[123]45	[23]145	[23]451	[234]51	32145	34521	[3245]1	[45]321	43215	43521	54321
12345	1	1	1	1	1	1	1	1	1	3	1	1
[123]45	1	2	3	3	3	3	3	3	3	3	3	3
[23]145	1	3	2	2	2	3	3	3	3	3	3	3
[23]451	1	3	2	2	2	3	3	3	3	3	3	3
[234]51	1	3	2	2	4	3	3	4	4	4	4	4
32145	1	3	3	3	3	3	3	3	3	3	3	3
34521	1	3	3	3	3	3	3	3	3	3	3	3
[3245]1	1	3	3	3	4	3	3	2	4	4	4	5
[45]321	1	3	3	3	4	3	3	4	4	4	4	5
43215	1	3	3	3	4	3	3	4	4	4	4	4
43521	1	3	3	3	4	3	3	4	4	4	4	4
54321	1	3	3	3	4	3	3	5	5	4	4	5

Table 2

Now, we present the notion of generalized uncompromisingness for RSCFs. It says that whenever an agent unilaterally moves his/her plateau in some direction, (i) if, for an alternative x, both the plateaus lie either strictly on the right of it or strictly on the left of it, then the probability of x does not change, and (ii) the probability of an interval $[x,m]$, where x lies in exactly one of the two plateaus, will weakly increase. In our formal definition, we present (i) by means of probabilities of sets of the form $[x,m]$, one can verify that it means exactly what we have explained above.

Definition 4.6. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies generalized uncompromisingness if for all $R_i, R_i' \in \mathscr{D}$, all $R_{N\setminus i} \in \mathscr{D}^{n-1}$, and all $x \in A$, we have

(i) $[x \leq \min\{\tau^-(R_i), \tau^-(R_i')\}]$ or $[\max\{\tau^+(R_i), \tau^+(R_i')\} < x]$ implies $\varphi_{[x,m]}(R_i', R_{N\setminus i}) = \varphi_{[x,m]}(R_i, R_{N\setminus i})$, and

(ii) $[\tau^+(R_i) < x \leq \tau^+(R_i')]$ or $[\tau^-(R_i) < x \leq \tau^-(R_i')]$ implies $\varphi_{[x,m]}(R_i', R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i, R_{N\setminus i})$.

REMARK 4.1. Note that as we have mentioned in the introduction of Definition 4.5, part (i) of generalized uncompromisingness implies that for all $(R_i, R_{N\setminus i}) \in \mathscr{D}^n$, all $i \in N$, and all $R_i' \in \mathscr{D}$, we have $\varphi_x(R_i, R_{N\setminus i}) = \varphi_x(R_i', R_{N\setminus i})$ for all $x \in A$ such that either $x < \{\tau^-(R_i), \tau^-(R_i')\}$ or $x > \max\{\tau^+(R_i), \tau^+(R_i')\}$.

REMARK 4.2. It is worth noting that the notion of generalized uncompromisingness coincides with that of uncompromisingness (Moulin (1980)) if we assume $\tau^+(R) = \tau^-(R)$ for all $R \in \mathscr{D}$, that is, if preferences are single-peaked.

REMARK 4.3. By considering R_i and R_i' such that $\tau^-(R_i) = \tau^-(R_i')$ and $\tau^+(R_i) = \tau^+(R_i')$, it follows that generalized uncompromisingness implies almost plateau-onlyness.

We illustrate the notion of generalized uncompromisingness by means of the following example. It is worth mentioning that, although the RSCF in the following example is chosen to be unanimous, the same is not implied by generalized uncompromisingness.

Example 4.3. Let the set of alternatives, agents, and admissible preferences be the same as in Example 4.2. In Table 3, we present an RSCF, say φ, that satisfies generalized uncompromisingness. To see that φ satisfies part (i) of generalized uncompromisingness, consider, for instance, the preference profiles $(12345, [123]45)$ and $(12345, [234]51)$. Note that agent 2 changes his/her plateau from $[123]$ to $[234]$ from the former preference profile to the latter. As alternative 5 lies strictly to the right of both the plateaus $[123]$ and $[234]$, $\varphi_5(12345, [123]45) = \varphi_5(12345, [234]51) = 0$. To see that φ satisfies part (ii) of generalized uncompromisingness, consider, for instance, the preference profiles $([2345]1, [234]51)$ and $([2345]1, [45]321)$. Alternative 3 lies (weakly) to the right of the plateau $[234]$ and strictly to the left of the plateau $[45]$. As required by (ii), $\varphi_{[3,5]}([2345]1, [234]51) < \varphi_{[3,5]}([2345]1, [45]321)$. It is worth mentioning that although the RSCF in this example is chosen to be unanimous, unanimity is not implied by generalized uncompromisingness. Later, we will make a formal remark to emphasize this fact.

1 \ 2	12345	[123]45	[23]145	[23]451	[234]51	32145	34521	[2345]1	[45]321	43215	43521	54321
12345	(1,0,0,0,0)	(1,0,0,0,0)	(0.3,0.7,0,0,0)	(0.3,0.7,0,0,0)	(0.3,0.7,0,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)	(0.3,0.3,0.4,0,0)
[123]45	(1,0,0,0,0)	(0.3,0.4,0.3,0,0)	(0,0.5,0.5,0,0)	(0,0.5,0.5,0,0)	(0,0.5,0.5,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0.6,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)
[23]145	(0.3,0.7,0,0,0)	(0,0.5,0.5,0,0)	(0,0.8,0.2,0,0)	(0,0.7,0.3,0,0)	(0,0.6,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)
[23]451	(0.3,0.7,0,0,0)	(0,0.5,0.5,0,0)	(0,0.7,0.3,0,0)	(0,0.4,0.6,0,0)	(0,0.5,0.5,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)
[234]51	(0.3,0.7,0,0,0)	(0,0.5,0.5,0,0)	(0,0.6,0.4,0,0)	(0,0.5,0.5,0,0)	(0,0.2,0.3,0.5,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0.4,0.4,0.2,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)
32145	(0.3,0.3,0.4,0,0)	(0,0.5,0.5,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)
34521	(0.3,0.3,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)
[2345]1	(0.3,0.3,0.4,0,0)	(0,0.6,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0.4,0.4,0.2,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0.2,0.3,0.3,0.2)	(0,0,0,0.5,0.5)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,0,1)
[45]321	(0.3,0.3,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,0.5,0.5)	(0,0,0,0.6,0.4)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,0,1)
43215	(0.3,0.3,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)
43521	(0.3,0.3,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,1,0)
54321	(0.3,0.3,0.4,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,0,0,1)	(0,0,0,0,1)	(0,0,0,1,0)	(0,0,0,1,0)	(0,0,0,0,1)

Table 3

□

We are now ready to present the main theorem of this section. It provides a characterization of unanimous and strategy-proof RSCFs by saying that a unanimous RSCF is strategy-proof if and only if it satisfies generalized uncompromisingness.

Theorem 4.3. *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathscr{D}$ be a (κ_1, κ_2)-single-plateaued domain. If $\varphi : \mathscr{D}^n \to \Delta A$ satisfies generalized uncompromisingness, then it is strategy-proof.*

The proof of the theorem is relegated to Appendix D.

Theorem 4.4. *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathscr{D}$ be a (κ_1, κ_2)-single-plateaued domain. Suppose $\varphi : \mathscr{D}^n \to \Delta A$ is unanimous. Then, φ is strategy-proof if and only if it satisfies generalized uncompromisingness.*

The proof of the theorem is relegated to Appendix E.

4.4 A FUNCTIONAL FORM CHARACTERIZATION OF THE UNANIMOUS AND STRATEGY-PROOF RULES FOR THE CASE OF TWO AGENTS

In this section, we consider the case where there are exactly two agents. We provide a complete characterization of the unanimous and strategy-proof rules in this scenario.

In what follows, we introduce the notion of random plateau rules. These rules are based on two parameters β_1 and β_2. Both these parameters represent some probability distributions over the set of alternatives. For instance, if the set of alternatives is $\{1, \ldots, 5\}$, then possible values of β_1 and β_2 are $(0.1, 0.2, 0.2, 0.3, 0.2)$ and $(0.4, 0, 0.1, 0.2, 0.3)$, respectively. Note that β_1 and β_2 are independent of each other.

Now, we explain how the the outcome of a random plateau rule φ is determined based on the parameter values β_1 and β_2. For any preference profile where $\tau(R_1) \cap \tau(R_2) \neq \emptyset$, define the outcome as an arbitrary probability distribution over $\tau(R_1) \cap \tau(R_2)$. Consider a preference profile where $\tau(R_1) \cap \tau(R_2) = \emptyset$. Suppose $\tau^+(R_1) < \tau^-(R_2)$. Consider an alternative x. If $x < \tau^+(R_1)$ or $\tau^-(R_2) < x$, then define $\varphi_x(R_N) = 0$. If $\tau^+(R_1) < x < \tau^-(R_2)$, define $\varphi_x(R_N) = \beta_2(x)$. Finally, if $x = \tau^+(R_1)$, then $\varphi_x(R_N) = \beta_2[1,x]$, and if $x = \tau^-(R_2)$, then $\varphi_x(R_N) = \beta_2[x,m]$. For the case where $\tau^+(R_2) < \tau^-(R_1)$, we use the probability distribution given by β_1 in place of β_2 to determine the outcomes.

For an example of a random plateau rule, consider $\beta_1 = (0.1, 0.2, 0.2, 0.3, 0.2)$ and $\beta_2 = (0.4, 0, 0.1, 0.2, 0.3)$. Let φ be a random plateau rule with respect to (β_1, β_2). In Table 4 , we provide the values of φ at some preference profiles.

R_N	$\varphi_1(R_N)$	$\varphi_2(R_N)$	$\varphi_3(R_N)$	$\varphi_4(R_N)$	$\varphi_5(R_N)$
$([12]345, 54321)$	0	0.4	0.1	0.2	0.3
$([234]15, [34]521)$	0	0	0.4	0.6	0
$([34]251, [12]345)$	0	0	0.3	0.7	0
$([543]21, 23451)$	0	0	0.3	0.7	0
$(54321, [23]145)$	0	0	0.5	0.3	0.2

Table 4

Below, we provide a formal definition of these rules.

Definition 4.7. An RSCF $\varphi : \mathscr{D}^2 \to \Delta A$ is called a random plateau rule with respect to (β_1, β_2), where $\beta_i \in \Delta A$ for all $i \in N$, if for all $R_N \in \mathscr{D}^2$ and all $x \in A$, we have

(i) $\tau(R_1) \cap \tau(R_2) \neq \emptyset$ implies $\varphi(R_N)$ is an arbitrary probability distribution over $\tau(R_1) \cap \tau(R_2)$,

(ii) $\tau^+(R_1) < \tau^-(R_2)$ implies $\varphi_{[x,m]}(R_N) = \beta_2[x,m]$ for all $\tau^+(R_1) < x \leq \tau^-(R_2)$ and $\varphi_x(R_N) = 0$ for all $x \notin [\tau^+(R_1), \tau^-(R_2)]$, and

(iii) $\tau^+(R_2) < \tau^-(R_1)$ implies $\varphi_{[x,m]}(R_N) = \beta_1[x,m]$ for all $\tau^+(R_2) < x \leq \tau^-(R_1)$ and $\varphi_x(R_N) = 0$ for all $x \notin [\tau^+(R_2), \tau^-(R_1)]$.

Now, we present the main result of this section. It characterizes all unanimous and strategy-proof rules on (κ_1, κ_2)-single-plateaued domains for two agents.

Theorem 4.5. *Let $N = \{1,2\}$ and $1 \leq \kappa_1 \leq \kappa_2 \leq m$. Suppose $\mathscr{D}$ is a (κ_1, κ_2)-single-plateaued domain. Then, an RSCF $\varphi : \mathscr{D}^2 \to \Delta A$ is unanimous and strategy-proof if and only if it is a random plateau rule.*

The proof of the theorem is relegated to Appendix F.

4.5 A CLASS OF UNANIMOUS AND STRATEGY-PROOF RULES

In this section, we present a large class of RSCFs that are unanimous and strategy-proof. These RSCFs are extension of random plateau rules for arbitrary number of agents. Like random plateau rules, these rules too are based on a class of parameters that we call monotonic.

Definition 4.8. A collection $(\beta_S)_{S\subseteq N}$, where $\beta_S \in \Delta A$ for all $S \subseteq N$, is called monotonic if for all $\emptyset \subseteq S \subset T \subseteq N$ and all $x \in A$, we have $\beta_S[x,m] \leq \beta_T[x,m]$.

Now, we introduce the notion of k-random plateau rules with respect to a class of monotonic parameters. We use the following notation: for a preference profile $R_N \in \mathscr{D}^n$, we denote by $I(R_N)$ the minimal interval such that $I(R_N) \cap \tau(R_i) \neq \emptyset$ for all $i \in N$, that is, the minimal interval that contains some top-ranked alternative of each agent.

Definition 4.9. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called a k-random plateau rule for some $k \in \{0,\dots,m\}$ with respect to a collection of monotonic parameters $(\beta_S)_{S\subseteq N}$ if for all $R_N \in \mathscr{D}^n$ and all $x \in A$, we have

(i) $\varphi_x(R_N) = 0$ for all $x \notin I(R_N)$, and

(ii) $\cap_{i\in N}\tau(R_i) = \emptyset$ implies that for all $x \in A$, $\varphi_{[x,m]}(R_N) = \beta_S[x,m]$, where $S \subseteq N$ is such that $i \in S$ if and only if $\tau^-(R_i) \geq x-k$ and $\tau^+(R_i) \geq x$

Note that one can construct a large class of k-random plateau rules by varying the values of βs and ks.

Next, we present the main theorem of this section.

Theorem 4.6. *Let $\mathscr{D}$ be a (κ_1,κ_2)-single-plateaued domain for some $1 \leq \kappa_1 \leq \kappa_2 \leq m$. Suppose $k \in \{0,\dots,\kappa_1-1\}$ and $(\beta_S)_{S\subseteq N}$ is a collection of monotonic parameters. Then, the k-random plateau rule $\varphi : \mathscr{D}^n \to \Delta A$ is unanimous and strategy-proof.*

The proof of the theorem is relegated to Appendix G.

In Section 4.6, we provide a characterization of unanimous, anonymous, plateau-only, and strategy-proof rules on a particular class of single-plateaued domains. It will be clear from that result the there are unanimous and strategy-proof rules other than k-random plateau rules on a single-plateaued domain.

4.6 A FUNCTIONAL FORM CHARACTERIZATION OF ANONYMOUS, PLATEAU-ONLY, AND STRATEGY-PROOF RULES

In this section, we provide a functional form characterization of anonymous, plateau-only, and strategy-proof RSCFs on a class of single-plateaued domains. By Theorem 4.2, every unanimous and strategy-proof

RSCF is almost plateau-only. We strengthen almost plateau-onlyness by plateau-onlyness in the interest of tractability. We also add anonymity for the same reason. Our characterization requires too many parameters even with these additional assumptions. Nevertheless, this characterization result can be extended by dropping anonymity in the same way as median rules are extended to min-max rules in the context of single-peaked domains (Moulin (1980)) . To replace plateau-onlyness by almost plateau-onlyness (which is implied by unanimity and strategy-proofness), one would require many more parameters, which, we think, will be too technical for its practical use.

In what follows, we present a collection of parameters that we require for the description of our RSCFs. Let $\kappa \in \{1,\ldots,m\}$. In our subsequent discussion, this κ is going to represent the size of the plateau of a single-plateaued preference. Given a κ, by $\underset{\sim}{n}$ we denote a vector $(n_0,n_1,\ldots,n_{\kappa-1})$ such that $0 \leq n_{\kappa-1} \leq \cdots \leq n_1 \leq n_0 \leq n$. For instance, if $\kappa = 3$ and $n = 4$, then an example of $\underset{\sim}{n}$ would be $(3,3,1)$. Let $\underset{\sim}{N}$ be the collection of all such vectors. Consider a table of order $|\underset{\sim}{N}| \times |\{2,\ldots,m\}|$ where the rows are indexed by the vectors $\underset{\sim}{n}$ in $\underset{\sim}{N}$ and columns are indexed by the alternatives in $\{2,\ldots,m\}$. See Table 6, for an example of such a table when $\kappa = 3$, $n = 2$, and $m = 5$.

Now, we identify some cells of the table described above for which we will define the values of our parameters. Let $\kappa \in \{1,\ldots,m\}$. Call the cell (corresponding to the position) $(\underset{\sim}{n},x)$ feasible for κ if $\underset{\sim}{n}_{\kappa-x} = n$ if $x \leq \kappa$ and $\underset{\sim}{n}_{m-x+1} = 0$ if $m-\kappa+1 < x$. For instance, if $\kappa = 3$, $n = 3$ and $m = 10$, then the following are some feasible cells: $((3,3,2),2),((3,3,1),2),((3,2,2),3)$, $((2,2,2),5),((3,1,1),7),((3,2,0),9),((2,0,0),10)$ and the following are some infeasible cells: $((3,1,0),2),((1,1,1),3),((3,3,3),9),((2,1,1),10)$. We denote by $\mathscr{F}(\kappa)$ the set of all feasible cells for κ.

We need the following terminologies to present some conditions on our parameters. For a vector $\underset{\sim}{n}$, we denote by $\underset{\sim}{n}^+$ the 'right-shifted' value of $\underset{\sim}{n}$, that is, $\underset{\sim}{n}^+_j = \underset{\sim}{n}_{j-1}$ if $1 \leq j \leq \kappa-1$. For instance, if $\underset{\sim}{n} = (5,3,2,2,1)$, then $\underset{\sim}{n}^+ = (\cdot,5,3,2,2)$. Here, any number that is weakly bigger than 5 (and weakly smaller than n) can appear at the position of the dot. For two vectors $\underset{\sim}{n}$ and $\underset{\sim}{n}'$, we write $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$ if there is $l \in \{0,\ldots,\kappa-1\}$ such that either $[\underset{\sim}{n}'_0 = \underset{\sim}{n}_0+1,\ldots,\underset{\sim}{n}'_l = \underset{\sim}{n}_l+1$ and $\underset{\sim}{n}'_{l+1} = \underset{\sim}{n}_{l+1},\ldots,\underset{\sim}{n}'_{\kappa-1} = \underset{\sim}{n}_{\kappa-1}]$ or $[\underset{\sim}{n}'_0 = \underset{\sim}{n}_0,\ldots,\underset{\sim}{n}'_l = \underset{\sim}{n}_l$ and $\underset{\sim}{n}'_{l+1} = \underset{\sim}{n}_{l+1}+1,\ldots,\underset{\sim}{n}'_{\kappa-1} = \underset{\sim}{n}_{\kappa-1}+1]$. In Table 5, we present some values of $\underset{\sim}{n}$ and $\underset{\sim}{n}'$ Note that when $\underset{\sim}{n}$ is $(3,2,2,1,0)$, the first, second and third components of $\underset{\sim}{n}$ are increased by 1, respectively, and the remaining are left unchanged. When $\underset{\sim}{n}$ is $(4,3,2,0,0)$, then the last and second-last components of $\underset{\sim}{n}$ are increased by 1 and the remaining are left unchanged.

Definition 4.10. Let $\kappa \in \{1,\ldots,m\}$. A collection $\{\beta(\underset{\sim}{n},x)_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ of numbers in $[0,1]$ is called plateau parameters for κ if for all $\underset{\sim}{n}$,

(i) $\beta(\underset{\sim}{n},x) \leq \beta(\underset{\sim}{n}^+,x-1)$, and

(ii) $\beta(\underset{\sim}{n},x) \leq \beta(\underset{\sim}{n} \oplus 1,x)$.

In Table 6, we present a collection of plateau parameters for the case when $\kappa = 3$, $n = 2$, and $m = 5$.

$\underset{\sim}{n}$	$\underset{\sim}{n}'$
$(3,2,2,1,0)$	$(4,3,3,1,0)$
$(4,3,2,0,0)$	$(4,3,2,1,1)$

Table 5

Example 4.4. Let $n = 2$, $\kappa = 3$ and $A = \{1,2,3,4,5\}$. Here

$$\begin{aligned}\mathscr{F}(3) = \{&((2,2,0),2),((2,2,1),2),((2,2,2),2),\\ &((2,0,0),3),((2,1,0),3),((2,2,0),3),((2,1,1),3),((2,2,1),3),((2,2,2),3),\\ &((0,0,0),4),((1,0,0),4),((1,1,0),4),((2,0,0),4),((2,1,0),4),((2,2,0),4),((1,1,1),4),\\ &((0,0,0),5),((1,0,0),5),((2,0,0),5)\}.\end{aligned}$$

Let $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)} \in \mathscr{F}(3)\}$ be a collection of plateau parameters as given in Definition 4.10. Consider $x = 3$, $\underset{\sim}{n} = (2,0,0)$ and $\underset{\sim}{n}' = (2,2,0)$. By Condition (i) in Definition 4.10, $0 \leq \beta((2,0,0),3) \leq \beta((2,2,0),2) \leq 1$. Take $x = 4$, $\underset{\sim}{n} = (0,0,0)$ and $\underset{\sim}{n}' = (1,1,0)$. By Condition (ii) of Definition 4.10, we must have $0 \leq \beta((0,0,0),4) \leq \beta((1,1,0),4) \leq 1$. Now, take $x = 4$, $\underset{\sim}{n} = (1,1,0)$ and $\underset{\sim}{n}' = (1,1,1)$. As βs are plateau parameters, by Condition (ii) of Definition 4.10, $0 \leq \beta((1,1,0),4) \leq \beta((1,1,1),4) \leq 1$. Table 6 provides an example of plateau parameters.

□

Definition 4.11. Let $\kappa \in \{1,\ldots,m\}$. A collection of plateau parameters $\{\beta(\underset{\sim}{n},x)_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ is called unanimous if $\beta(\underset{\sim}{n},x) = 1$ whenever $\underset{\sim}{n}_0 = n$ and $\underset{\sim}{n}_{\kappa-1} > 0$, and $\beta(\underset{\sim}{n},x) = 0$ whenever $\underset{\sim}{n}_0 < n$ and $\underset{\sim}{n}_{\kappa-1} = 0$.

In Table 7, we provide a collection of unanimous plateau parameters.

In Example 4.4, for $\{\beta(\underset{\sim}{n},x)_{(\underset{\sim}{n},x)\in\mathscr{F}(3)}\}$ to be unanimous plateau parameters we must have $\beta((2,2,1),2) = \beta((2,2,2),2) = \beta((2,1,1),3) = \beta((2,2,1),3) = \beta((2,2,2),3) = 1$ and $\beta((0,0,0),4) = \beta((1,0,0),4) = \beta((1,1,0),4) = \beta((0,0,0),5) = \beta((1,0,0),5) = 0$.

In what follows, we present the notion of κ-plateaued rules. We need the following terminology for our presentation. For an alternative $x \in A$, a number $l \in \{0,\ldots,\kappa-1\}$, and a preference profile $R_N \in \mathscr{D}^n$,

(n_0,n_1,n_2)	2	3	4	5
(0,0,0)			.4	.3
(1,0,0)			.5	.4
(1,1,0)			.45	
(1,1,1)				
(2,0,0)		.6	.6	.45
(2,1,0)		.55	.55	
(2,1,1)		.55		
(2,2,0)	.7	.6	.5	
(2,2,1)	.8	.7		
(2,2,2)	.9	.8		

Table 6

(n_0,n_1,n_2)	2	3	4	5
(0,0,0)			0	0
(1,0,0)			0	0
(1,1,0)			0	
(1,1,1)				
(2,0,0)		0.6	.6	.45
(2,1,0)		.55	.55	
(2,1,1)		1		
(2,2,0)	0.7	.6	.5	
(2,2,1)	1	1		
(2,2,2)	1	1		

Table 7

let $n_l^x(R_N) = |\{i \in N \mid \tau^+(R_i) \geq x+l\}|$ be the set of agents whose right-end point of the plateau at R_N is (weakly) on the right of $x+l$.

Definition 4.12. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called a κ-plateaued rule for if there is a collection of plateau parameters $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ for κ such that for all $R_N \in \mathscr{D}^n$, $\varphi_{[x,m]}(R_N) = \beta(\underset{\sim}{n},x)$, where $\underset{\sim}{n_l} = n_l^x(R_N)$ for all $0 \leq l \leq \kappa - 1$.

In Example 4.5, we present a κ-plateaued rule .

Example 4.5. Let $n=2$, $\kappa = 3$ and $A = \{1,2,3,4,5\}$. Let φ be a 3-plateaued rule with respect to a collection $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(3)}\}$ as given in Example 4.4. Consider the preference profile $R_N \in \mathscr{D}^n$ where $\tau(R_1) = [1,3]$ and $\tau(R_2) = [3,5]$. Take $x=5$. Note that $n^5(R_N) = (1,0,0)$. Thus, $\varphi_5(R_N) = \beta(5,(1,0,0)) = 0.4$. Similarly, $n^4(R_N) = (1,1,0)$, $n^3(R_N) = (2,1,1)$ and $n^2(R_N) = (2,2,1)$ (refer to Figure 1). As φ is a 3-plateaued rule , $\varphi_{[4,5]}(R_N) = 0.45$, $\varphi_{[3,5]}(R_N) = 0.55$, $\varphi_{[2,5]}(R_N) = 0.8$ and

$\varphi_{[1,5]}(R_N) = 1$. Thus $\varphi_5(R_N) = 0.4$, $\varphi_4(R_N) = 0.05$, $\varphi_3(R_N) = 0.1$, $\varphi_2(R_N) = 0.25$ and $\varphi_1(R_N) = 0.2$. The complete rule is given in Table 8. □

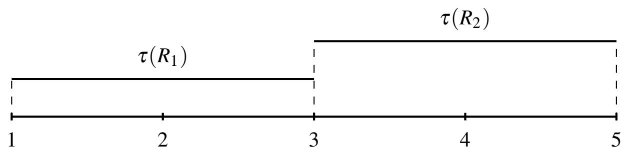

Figure 1

	[1,3]	[2,4]	[3,5]
[1,3]	(0.3,0.1,0.2,0.1,0.3)	(0.2,0.25,0.05,0.2,0.3)	(0.2,0.25,0.1,0.05,0.4)
[2,4]	(0.2,0.25,0.05,0.2,0.3)	(0.1,0.3,0,0.3,0.3)	(0.1,0.2,0.15,0.15,0.4)
[3,5]	(0.2,0.25,0.1,0.05,0.4)	(0.1,0.2,0.15,0.15,0.4)	(0.1,0.1,0.3,0.05,0.45)

Table 8

Now, we are ready to present the main results of this section. For ease of presentation, we call a (κ,κ)-singe-plateaued domain a κ-single-plateaued domain. Note that if $\kappa = 1$, then a $\mathscr{D}$ domain is the single-peaked domain. Theorem 4.7 characterizes all anonymous, plateau-only, and strategy-proof RSCFs on a κ-single-plateaued domain.

Theorem 4.7. *Let $\kappa \in \{1,\dots,m\}$ and $\mathscr{D}$ be a κ-single-plateaued domain. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is anonymous, plateau-only, and strategy-proof if and only if it is κ-plateaued rule .*

The proof of the theorem is relegated to Appendix H.

Corollary 4.1. *An RSCF is unanimous, anonymous, plateau-only, and strategy-proof if and only if it is a κ-plateaued rule with respect to some unanimous plateau parameters.*

The proof of the corollary is relegated to Appendix I.

Our next theorem says that a κ-plateaued rule is strategy-proof (together with being anonymous and plateau-only) on any single-plateaued domain such that the size of the plateau for any preference in it is at least κ.

Theorem 4.8. *Let $\kappa \in \{1,\dots,m\}$ and let $\mathscr{D}$ be a $(\kappa,\hat{\kappa})$-single-plateaued domain for some (arbitrary) $\hat{\kappa} \geq \kappa$. Suppose $\varphi : \mathscr{D}^n \to \Delta A$ is a κ-plateaued rule . Then, φ is anonymous, plateau-only, and strategy-proof.*

The proof of the theorem is relegated to Appendix J.

5. Conclusion

In this paper we study the structure of unanimous (or Pareto optimality) and strategy-proof random social choice functions when weak preferences are admissible. 3.1 of this paper shows that under some minimal richness condition on the weak single-peaked domain an RSCF is Pareto optimal and strategy-proof if and only if it is an extreme PFBR. An interesting application of this result is the single-peaked domain with outside options (see Cantala (2004)). Theorem Theorem 4.1 shows that on a single-plateaued domain under strategy-proofness, unanimity and Pareto optimality are equivalent for RSCFs. Theorem 4.2 shows that any unanimous and strategy-proof RSCF is almost plateau-only.

Next, in Theorems 4.3 and 4.4, we provide an axiomatic characterization of the unanimous and strategy-proof RSCFs. We show that an RSCF is unanimous and strategy-proof if and only if it satisfies a generalized version of uncompromisingness. Uncompromisingness says that as long as the plateau of an individual stay on one side of an alternative, the probability of that alternative cannot be changed. Generalized uncompromisingness additionally imposes some restriction on how the probability of an alternative can change when the plateau of an individual crosses it.

Finally, we proceed to present a functional form presentation of RSCFs. We provide a functional form characterization of the unanimous and strategy-proof rules on (κ_1, κ_2)-single-plateaued domains for two players. We also provide a class of unanimous and strategy-proof RSCFs on (κ_1, κ_2)-single-plateaued domains for more than two players. In 4.7 we strengthen almost plateau-onlyness by plateau-onlyness and provide a functional form characterization of the plateau-only, anonymous, and strategy-proof RSCFs.

A. Proof of Theorem 3.1

Proof. ("If" part) We show that an extreme PFBR is Pareto optimal and strategy-proof. Since an alternative receives positive probability in an extreme PFBR at a profile if and only if the alternative is top-ranked by at least one agent, it follows that the outcome of an extreme PFBR at any profile cannot be Pareto dominated, which implies that such a rule is Pareto optimal. Let φ be an extreme PFBR. To show strategy-proofness, let us assume for contradiction that $\sum_{y\in U(x,R_i)} \varphi_y(R_i', R_{N\setminus i}) > \sum_{y\in U(x,R_i)} \varphi_y(R_i, R_{N\setminus i})$ for some R_N and R_i'. Let $\widehat{P}_N \in \text{strict}(\mathscr{D}_N)$ and $\bar{P}_i \in \text{strict}(\mathscr{D}_i)$ be such that $\tau(\widehat{P}_i) = \tau(R_i)$ for all $i \in N$, $\tau(\bar{P}_i) = \tau(R_i')$, and $U(x, R_i)$ is an upper contour set of $\widehat{P}_i$. By the definition of an extreme PFBR, we have $\varphi(\widehat{P}_N) = \varphi(R_N)$ and $\varphi(\bar{P}_i, \widehat{P}_{N\setminus i}) = \varphi(R_i', R_{N\setminus i})$. This implies $\sum_{y\in U(x,R_i)} \varphi_y(\bar{P}_i, \widehat{P}_{N\setminus i}) > \sum_{y\in U(x,R_i)} \varphi_y(\widehat{P}_N)$, which in turn means that the restriction of φ on $\text{strict}(\mathscr{D}_N)$ is manipulable. However, this is a contradiction since the restriction of φ on strict($\mathscr{D}_N$) is a PFBR which is known to be strategy-proof (see Ehlers et al. (2002).

("Only-if" part) A profile $R_N \in \mathscr{D}_N$ is called a boundary profile if $\tau(R_i) \in \{1,m\}$ for all $i \in N$, that is, the top-ranked alternative of any preference in such a profile lies on the boundary 1 or m of the set of alternatives. We make extensive use of two particular types of strict single-peaked preferences in our proofs: a single-peaked preference P is called left (or right) if for all $x < \tau(P)$ and all $y > \tau(P)$, we have xPy (or yPx).

Lemma A.1. *Let $i \in N$, $P_i \in strict(\mathscr{D}_i)$, and $R_{N\setminus i} \in \mathscr{D}_{N\setminus i}$. Suppose $r < \tau(P_i) < s$ are such that sP_ir. Then,*

(i) there exists $\bar{P}_i \in \mathscr{D}_i$ with $\bar{P}_i \equiv \tau(P_i)\cdots sr\cdots$ such that $\varphi_s(P_i,R_{N\setminus i}) \geq \varphi_s(\bar{P}_i,R_{N\setminus i})$, and

(ii) $\bar{\bar{P}}_i \in \mathscr{D}_i$ with $\bar{\bar{P}}_i \equiv \tau(P_i)\cdots rs\cdots$ such that $\varphi_s(P_i,R_{N\setminus i}) > \varphi_s(\bar{\bar{P}}_i,R_{N\setminus i})$ implies $\varphi_k(P_i,R_{N\setminus i}) < \varphi_k(\bar{\bar{P}}_i,R_{N\setminus i})$ for some $k \in [r,\tau(P_i))$.

Proof. By the definition of single-peakedness, if sP_iaP_ir for some $a \in A$, then either $a > s$ or $a \in (r,\tau(P_i))$. Consider the strict single-peaked preference $\hat{P}_i \in$ strict($\mathscr{D}_i$) such that $\tau(\hat{P}_i) = \tau(P_i)$, $U(s,P_i) = U(s,\hat{P}_i)$ and $s\hat{P}_ia\hat{P}_ir$ for some $a \in A$ if and only if $a \in (r,\tau(P_i)) \setminus U(s,P_i)$. Existence of such a preference is guaranteed by minimal richness. By strategy-proofness, $\varphi_s(P_i,R_{N\setminus i}) = \varphi_s(\hat{P}_i,R_{N\setminus i})$. By the definition of $\hat{P}_i$, there exists $l \geq 0$ such that $\hat{P}_i \equiv \cdots s(r+l)(r+l-1)\cdots(r+1)r\cdots$. Let $\hat{\hat{P}}_i$ be obtained by swapping the alternatives s and $(r+l)$ at $\hat{P}_i$. Thus, $\hat{\hat{P}}_i \equiv \cdots(r+l)s(r+l-1)\cdots(r+1)r\cdots$. Note that $\hat{\hat{P}}_i$ is strict single-peaked. By straightforward application of strategy-proofness, $\varphi_s(\hat{P}_i,R_{N\setminus i}) \geq \varphi_s(\hat{\hat{P}}_i,R_{N\setminus i})$. Continuing in this manner, we can arrive at a preference $\bar{P}_i$ such that $\bar{P}_i \equiv \tau(P_i)\cdots sr\cdots$ and $\varphi_s(P_i,R_{N\setminus i}) \geq \varphi_s(\bar{P}_i,R_{N\setminus i})$. This completes the proof of part (i) of the lemma.

Let $\bar{\bar{P}}_i$ be the strict single-peaked preference obtained by swapping s and r at $\bar{P}_i$. Since $\varphi_s(P_i,R_{N\setminus i}) = \varphi_s(\hat{P}_i,R_{N\setminus i})$ and we have arrived at the preference $\bar{\bar{P}}_i$ from $\hat{P}_i$ by a sequence of swaps between s and some alternatives in the set $\{r,\ldots,r+l\}$, if $\varphi_s(\bar{\bar{P}},R_{N\setminus i}) < \varphi_s(P_i,R_{N\setminus i})$, then there must exist some $a \in \{r,\ldots,r+l\}$ such that $\varphi_a(\bar{P}_i,R_{N\setminus i}) > \varphi_a(P_i,R_{N\setminus i})$. This completes the proof of part (ii) of the lemma. ■

We prove the "only-if" part of the theorem in two steps. In the first step, we show that every Pareto optimal and strategy-proof RSCF on the domain strict($\mathscr{D}_N$) behaves like an extreme PFBR on the set of boundary profiles. In the next step, we show that the same happens on every profile.

Step 1. Let φ be a Pareto optimal and strategy-proof RSCF. We show that φ is an extreme PFBR. The following claim says that only the boundary alternatives 1 and m can get positive probability at boundary profiles.

Claim A.1. $\varphi_x(R_N) = 0$ *for all $x \in \{2,\ldots,m-1\}$ and all boundary profiles $R_N \in \mathscr{D}_N$.*

Proof of Claim. Assume for contradiction $\varphi_x(R_N) > 0$ for some $x \in \{2,\dots,m-1\}$ and for some boundary profile $R_N \in \mathscr{D}_N$. For each $R_i \in \mathscr{D}_i$, let $\bar{R}_i$ be the dichotomous preference with $\tau(\bar{R}_i) = \tau(R_i)$. Note that such preferences exist as the domain is minimally rich. By Pareto optimality, $\varphi_x(\bar{R}_N) = 0$ for all $x \in \{2,\dots,m-1\}$. For all $x \in \{1,m\}$, let N_x be the set of agents i whose top ranked alternative at $\bar{R}_i$ is x, that is $N_x = \{i \in N \mid \tau(\bar{R}_i) = x\}$. Take $i \in N$. Consider the profile $(R_i, \bar{R}_{N\setminus i})$. By strategy-proofness, $\varphi_1(R_i, \bar{R}_{N\setminus i}) = \varphi_1(\bar{R}_i, \bar{R}_{N\setminus i})$. Also, by Pareto optimality, $\varphi_x(R_i, \bar{R}_{N\setminus i}) = 0$ for all $x \in \{2,\dots,m-1\}$. This is because if $\varphi_x(R_i, \bar{R}_{N\setminus i}) > 0$ for some $x \in \{2,\dots,m-1\}$, then shifting this probability to 1 will be a Pareto improvement. Thus, $\varphi(R_i, \bar{R}_{N\setminus i}) = \varphi(\bar{R}_i, \bar{R}_{N\setminus i})$. Applying this logic repeatedly for all agents in N_1, we obtain $\varphi(R_{N_1}, \bar{R}_{N\setminus N_1}) = \varphi(\bar{R}_{N_1}, \bar{R}_{N\setminus N_1})$. Now, consider $i \in N_m$. Since $\tau(\bar{R}_i) = \tau(R_i) = m$, by strategy-proofness, $\varphi_m(R_i, R_{N_1}, \bar{R}_{N\setminus N_1 \cup i}) = \varphi_m(\bar{R}_i, R_{N_1}, \bar{R}_{N\setminus N_1 \cup i})$. This, combined with the fact that $\varphi(R_{N_1}, \bar{R}_{N\setminus N_1}) = \varphi(\bar{R}_{N_1}, \bar{R}_{N\setminus N_1})$, yields

$$\varphi_m(R_i, R_{N_1}, \bar{R}_{N\setminus N_1 \cup i}) = \varphi_m(\bar{R}_N). \tag{1}$$

Applying this argument for all agents in $N_m \setminus \{i\}$, we obtain $\varphi_m(R_N) = \varphi_m(\bar{R}_N)$. Since $\varphi_x(R_N) > 0$ for some $x \in \{2,\dots,m-1\}$, this implies

$$\varphi_1(R_N) < \varphi_1(\bar{R}_N). \tag{2}$$

Note that we can arrive at the profile R_N in a symmetrically opposite way: by changing the preferences of agents in N_m from $\bar{R}_i$ to R_i first, and then changing the preferences of agents in N_1 from $\bar{R}_i$ to R_i. Therefore, by using the same argument as for obtaining (1), we can conclude that $\varphi_1(R_N) = \varphi_1(\bar{R}_N)$, which contradicts (2). □

To prove that φ is an extreme PFBR, it remains to show that φ is monotonic over the boundary profiles, that is, for all $R_N, R'_N \in \mathscr{D}_N$ with $S(m, R_N) \subseteq S(m, R'_N)$, we have $\varphi_m(R_N) \leq \varphi_m(R'_N)$. This follows by straightforward application of strategy-proofness.

Step 2. Let strict$(\mathscr{D}_i) \subset \mathscr{D}_i$ be the strict single-peaked domain contained in $\mathscr{D}_i$. Since φ is Pareto optimal and strategy-proof, it must be a PFBR on strict$(\mathscr{D}_N)$. By Step 1, it follows that φ is an extreme PFBR. It is sufficient to show that φ is tops-only on $\mathscr{D}_N$. We prove this by using induction on the number of agents in a preference profile having non-strict preferences. We begin with the base case where there exists exactly one agent having non-strict preference.

Base case: Let $P_N \in$ strict$(\mathscr{D}_N)$ be a strict single-peaked preference profile and let R_i be a non-strict preference with $\tau(R_i) = \tau(P_i)$. We show that $\varphi(P_N) = \varphi(R_i, P_{N\setminus i})$. Assume for contradiction that

$\varphi(P_N) \neq \varphi(R_i, P_{N\setminus i})$. Without loss of generality, let $s > \tau(P_i)$ be such that $\varphi_s(R_i, P_{N\setminus i}) > \varphi_s(P_i, P_{N\setminus i})$ and $\varphi_t(R_i, P_{N\setminus i}) \leq \varphi_t(P_i, P_{N\setminus i})$ for all $t \in [\tau(P_i), s)$.

Case B1. Suppose that there is no agent j such that $\tau(P_j) \in (\tau(P_i), s)$.

Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(\hat{P}_i) = \tau(R_i) = \tau(P_i)$. Since $(\hat{P}_i, P_{N\setminus i}) \in \text{strict}(\mathscr{D}_N)$, by only-topsness $\varphi_t(\hat{P}_i, P_{N\setminus i}) = 0$ for all $t \in (\tau(P_i), s)$. By strategy-proofness $\varphi_{U(s,\hat{P}_i)}(\hat{P}_i, P_{N\setminus i}) \geq \varphi_{U(s,\hat{P}_i)}(R_i, P_{N\setminus i})$ and $\varphi_{\tau(P_i)}(\hat{P}_i, P_{N\setminus i}) = \varphi_{\tau(P_i)}(R_i, P_{N\setminus i})$. Combining all these observations, it follows that

$$\varphi_s(\hat{P}_i, P_{N\setminus i}) \geq \varphi_s(R_i, P_{N\setminus i}). \tag{3}$$

Recall that

$$\varphi_s(R_i, P_{N\setminus i}) > \varphi_s(P_i, P_{N\setminus i}). \tag{4}$$

By (3) and (4) this implies

$$\varphi_s(\hat{P}_i, P_{N\setminus i}) > \varphi_s(P_i, P_{N\setminus i}). \tag{5}$$

Since $\tau(P_i) = \tau(\hat{P}_i)$, we have $\varphi_s(P_i, P_{N\setminus i}) = \varphi_s(\hat{P}_i, P_{N\setminus i})$, which contradicts (5).

Case B2. Suppose that Case 1 does not hold, that is, there are agents j such that $\tau(P_j) \in (\tau(P_i), s)$. Consider $j \in N$ such that $\tau(P_j) \in (\tau(P_i), s)$ and there does not exist $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$

Claim A.2. *$\varphi_s(R_i, \hat{P}_j, P_{N\setminus\{i,j\}}) = \varphi_s(R_i, P_j, P_{N\setminus\{i,j\}})$ for some $\hat{P}_j \in$ strict$(\mathscr{D}_j)$ with $\tau(\hat{P}_j) = \tau(P_i)$.*

Proof of Claim A.2

Case B2.1. Suppose $\tau(P_i)P_js$.

Since $\tau(P_i)P_js$, there exists $\hat{P}_j$ with $\tau(\hat{P}_j)$ such that $U(s, \hat{P}_j) = U(s, P_j)$. By strategy-proofness, $\varphi_s(R_i, \hat{P}_j, P_{N\setminus\{i,j\}}) = \varphi_s(R_i, P_j, P_{N\setminus\{i,j\}})$.

Case B2.2. Suppose $sP_j\tau(P_i)$.

In view of Case 1 it is sufficient to show $\varphi_s(R_i, P_j, P_{N\setminus\{i,j\}}) = \varphi_s(R_i, \bar{P}_j, P_{N\setminus\{i,j\}})$ for some $\bar{P}_j$ with $\tau(\bar{P}_j) = \tau(P_j)$ and $\tau(P_i)\bar{P}_js$.

Since $\tau(P_i) < \tau(P_j) < s$, by Lemma A.1, we can construct a preference $\tilde{P}_j \equiv \cdots s\tau(P_i) \cdots$ such that

$$\varphi_s(R_i, P_j, P_{N\setminus\{i,j\}}) \geq \varphi_s(R_i, \tilde{P}_j, P_{N\setminus\{i,j\}}). \tag{6}$$

Construct a preference $\tilde{\tilde{P}}_j$ by swapping s and $\tau(P_i)$ in $\tilde{P}_j$. By strategy-proofness,

$$\varphi_s(R_i, \tilde{P}_j, P_{N\setminus\{i,j\}}) \geq \varphi_s(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}). \tag{7}$$

Combining (6) and (7), $\varphi_s(R_i,P_j,P_{N\setminus\{i,j\}}) \geq \varphi_s(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$. If $\varphi_s(R_i,P_j,P_{N\setminus\{i,j\}}) = \varphi_s(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$, then we take $\bar{P}_j = \tilde{\tilde{P}}_j$. Suppose $\varphi_s(R_i,P_j,P_{N\setminus\{i,j\}}) > \varphi_s(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$. It follows from Lemma A.1 that there exists $r \in [\tau(P_i),\tau(P_j))$ such that $\varphi_r(R_i,P_j,P_{N\setminus\{i,j\}}) < \varphi_r(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$. Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(P_i) = \tau(\hat{P}_i)$. Since $(\hat{P}_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}}) \in \text{strict}(\mathscr{D}_N)$, by only-topsness and our assumption that there is no $l \in N$ such that $\tau(P_l) \in (\tau(P_i),\tau(P_j))$, we have $\varphi_t(\hat{P}_i,\tilde{\tilde{P}},P_{N\setminus\{i,j\}}) = 0$ for all $t \in (\tau(P_i),\tau(P_j))$. Consider $U(r,\hat{P}_i)$. By strategy-proofness, $\varphi_{U(r,\hat{P}_i)}(\hat{P}_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}}) \geq \varphi_{U(r,\hat{P}_i)}(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$. Because $\varphi_r(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}}) > 0$ and $\varphi_t(\hat{P}_i,\tilde{\tilde{P}},P_{N\setminus\{i,j\}}) = 0$ for all $t \in U(r,\hat{P}_i) \setminus \{\tau(P_i)\}$, this implies $\varphi_{\tau(P_i)}(\hat{P}_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}}) > \varphi_{\tau(P_i)}(R_i,\tilde{\tilde{P}}_j,P_{N\setminus\{i,j\}})$. Since $\tau(P_i) = \tau(\hat{P}_i) = \tau(R_i)$, this is a contradiction to strategy-proofness. □

Consider the profile $(R_i,\hat{P}_j,P_{N\setminus\{i,j\}})$. Let $l \in N$ be such that $\tau(P_l) \in (\tau(P_i),s)$ and $\tau(P_{\hat{l}}) \notin (\tau(P_i),\tau(P_l))$ for all $\hat{l} \in N$. By using similar logic as for Claim A.2, we can construct a preference $\hat{P}_l$ such that $\tau(\hat{P}_l) = \tau(P_l)$ and $\varphi_s(R_i,\hat{P}_j,P_{N\setminus\{i,j\}}) = \varphi_s(R_i,\hat{P}_j,\hat{P}_l,P_{N\setminus\{i,j,l\}})$. Continuing in this manner, we can construct a profile $(R_i,\hat{P}_{N\setminus i})$ such that $\tau(\hat{P}_j) = \tau(P_i)$ if $\tau(P_j) \in (\tau(P_i),s)$ and $\hat{P}_j = P_j$ if $\tau(P_j) \notin (\tau(P_i),s)$ and

$$\varphi_s(R_i,\hat{P}_{N\setminus i}) = \varphi_s(R_i,P_{N\setminus i}) \tag{8}$$

Note that by construction of $\hat{P}_{N\setminus i}$, there is no agent j such that $\tau(\hat{P}_j) \in (\tau(P_i),s)$. Thus, by applying the same logic as in Case B1, $\varphi_s(R_i,\hat{P}_{N\setminus i}) = \varphi_s(P_i,P_{N\setminus i})$. Combining this with (8), we have $\varphi_s(P_i,P_{N\setminus i}) = \varphi_s(R_i,P_{N\setminus i})$.

Induction step: Suppose that φ behaves like an extreme PFBR over all profiles at which at most k agents have non-strict preferences. We proceed to show that the same holds over all profiles where $k+1$ agents have non-strict preferences.

Consider $(R_S,P_{N\setminus S}) \in \mathscr{D}_N$ such that R_i is not strict for all $i \in S$, P_i is strict for all $i \in N \setminus S$ and $|S| = k+1$. Assume without loss of generality, agent $1 \in S$. Let P_1 be a strict preference with $\tau(P_1) = \tau(R_1)$. In view of our induction hypothesis, it is enough to show, $\varphi(R_S,P_{N\setminus S}) = \varphi(P_1,R_{S\setminus 1},P_{N\setminus S})$. Clearly, by strategy-proofness, $\varphi_{\tau(R_1)}(R_S,P_{N\setminus S}) = \varphi_{\tau(R_1)}(P_1,R_{S\setminus 1},P_{N\setminus S})$. Consider $l \in A$ such that $\tau(R_i) = l$ for some $i \in S \setminus 1$. We show that $\varphi_l(R_S,P_{N\setminus S}) = \varphi_l(P_1,R_{S\setminus 1},P_{N\setminus S})$. Let P_i be a strict preference with $\tau(P_i) = \tau(R_i)$. By strategy-proofness,

$$\varphi_l(P_i,R_{S\setminus i},P_{N\setminus S}) = \varphi_l(R_S,P_{N\setminus S}). \tag{9}$$

Since both the profiles $(P_i,R_{S\setminus i},P_{N\setminus S})$ and $(P_1,R_{S\setminus 1},P_{N\setminus S})$ have exactly k agents with non-strict preferences, by our induction hypothesis,

$$\varphi_l(P_i, R_{S\setminus i}, P_{N\setminus S}) = \varphi_l(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{10}$$

Combining (9) and (10), we have $\varphi_l(R_S, P_{N\setminus S}) = \varphi_l(P_1, R_{S\setminus 1}, P_{N\setminus S})$.

Next, we proceed to show $\varphi(R_S, P_{N\setminus S}) = \varphi(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Assume for contradiction that $\varphi(R_S, P_{N\setminus S}) \neq \varphi(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Without loss of generality $s > \tau(P_1)$ be such that $\varphi_s(R_S, P_{N\setminus S}) > \varphi_s(P_1, R_{S\setminus 1} P_{N\setminus S})$ and $\varphi_t(R_S, P_{N\setminus S}) \leq \varphi_t(P_1, R_{S\setminus 1} P_{N\setminus S})$ for all $t \in (\tau(P_1), s)$. Let $i \in S$ be such that there does not exist $l \in S$ such that $\tau(R_l) \in (\tau(R_i), s]$. The rest of the proof follows by using similar logic as for the base case, but for the sake of completeness, we present it.

Case I1. Suppose there is no agent j such that $\tau(R_j) \in (\tau(R_i), s)$.

Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(\hat{P}_i) = \tau(R_i) = \tau(P_i)$. Since $(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S})$ has exactly k agents with non-strict preferences, by our induction hypothesis $\varphi_t(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = 0$ for all $t \in (\tau(P_i), s)$. By strategy-proofness $\varphi_{U(s,\hat{P}_i)}(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) \geq \varphi_{U(s,\hat{P}_i)}(R_S, P_{N\setminus S})$ and $\varphi_{\tau(P_i)}(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = \varphi_{\tau(P_i)}(R_S, P_{N\setminus S})$. Combining all these observations,

$$\varphi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus i}) \geq \varphi_s(R_S, P_{N\setminus S}). \tag{11}$$

Recall that

$$\varphi_s(R_S, P_{N\setminus S}) > \varphi_s(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{12}$$

By (11) and (12) this implies

$$\varphi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) > \varphi_s(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{13}$$

Since at the profiles $(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S})$ and $(P_1, R_{S\setminus 1}, P_{N\setminus S})$, $\varphi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = \varphi_s(P_1, R_{S\setminus 1}, P_{N\setminus S})$, which contradicts (13).

Case I2. Suppose that Case 1 does not hold, that is, there are agents j such that $\tau(P_j) \in (\tau(P_i), s)$.

Consider $j \in N$ such that $\tau(P_j) \in (\tau(P_i), s)$ and there does not exist $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$.

Claim A.3. *$\varphi_s(R_S, \hat{P}_j, P_{N\setminus S\cup j}) = \varphi_s(R_S, P_j, P_{N\setminus S\cup j})$ for some $\hat{P}_j \in strict(\mathscr{D}_j)$ with $\tau(\hat{P}_j) = \tau(P_i)$.*

Proof of Claim A.3

Case I2.1. Suppose $\tau(P_i) P_j s$.

Since $\tau(P_i) P_j s$, there exists $\hat{P}_j$ with $\tau(\hat{P}_j)$ such that $U(s, \hat{P}_j) = U(s, P_j)$. By strategy-proofness, $\varphi_s(R_S, \hat{P}_j, P_{N\setminus S\cup j}) = \varphi_s(R_S, P_j, P_{N\setminus S\cup j})$.

Case I2.2. Suppose $sP_j\tau(P_i)$.

In view of Case 1 it is sufficient to show $\varphi_s(R_S,P_j,P_{N\setminus S\cup j}) = \varphi_s(R_S,\bar{P}_j,P_{N\setminus S\cup j})$ for some $\bar{P}_j$ with $\tau(\bar{P}_j) = \tau(P_j)$ and $\tau(P_i)\bar{P}_j s$.

Since $\tau(P_i) < \tau(P_j) < s$, by Lemma A.1, we can construct a preference $\tilde{P}_j \equiv \cdots s\tau(P_i)\cdots$ such that

$$\varphi_s(R_S,P_j,P_{N\setminus S\cup j}) \geq \varphi_s(R_S,\tilde{P}_j,P_{N\setminus S\cup j}). \tag{14}$$

Construct a preference $\tilde{\tilde{P}}_j$ by swapping s and $\tau(P_i)$ in $\tilde{P}_j$. By strategy-proofness,

$$\varphi_s(R_S,\tilde{P}_j,P_{N\setminus S\cup j}) \geq \varphi_s(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j}). \tag{15}$$

Combining (14) and (15), $\varphi_s(R_S,P_j,P_{N\setminus S\cup\{j\}}) \geq \varphi_s(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup\{j\}})$. If $\varphi_s(R_S,P_j,P_{N\setminus S\cup j}) = \varphi_s(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$, then we take $\bar{P}_j = \tilde{\tilde{P}}_j$. Suppose $\varphi_s(R_S,P_j,P_{N\setminus S\cup j}) > \varphi_s(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$. It follows from Lemma A.1 that there exists $r \in [\tau(P_i),\tau(P_j))$ such that $\varphi_r(R_S,P_j,P_{N\setminus S\cup j}) < \varphi_r(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$. Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(P_i) = \tau(\hat{P}_i)$. Since $(\hat{P}_i,R_{S\setminus i},\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$ has exactly k agents with non-strict preferences, by our induction hypothesis and assumption that there is no $l \in N$ such that $\tau(P_l) \in (\tau(P_i),\tau(P_j))$, we have $\varphi_t(\hat{P}_i,R_{S\setminus i},\tilde{\tilde{P}}_j,P_{N\setminus S\cup j}) = 0$ for all $t \in (\tau(P_i),\tau(P_j))$. Consider $U(r,\hat{P}_i)$. By strategy-proofness, $\varphi_{U(r,\hat{P}_i)}(\hat{P}_i,R_{S\setminus\{i\}},\tilde{\tilde{P}}_j,P_{N\setminus S\cup j}) \geq \varphi_{U(r,\hat{P}_i)}(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$. Because $\varphi_r(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j}) > 0$ and $\varphi_t(\hat{P}_i,R_{S\setminus i}\tilde{\tilde{P}},P_{N\setminus S\cup j}) = 0$ for all $t \in U(r,\hat{P}_i)\setminus\{\tau(P_i)\}$, this implies $\varphi_{\tau(P_i)}(\hat{P}_i,R_{S\setminus i},\tilde{\tilde{P}}_j,P_{N\setminus S\cup j}) > \varphi_{\tau(P_i)}(R_S,\tilde{\tilde{P}}_j,P_{N\setminus S\cup j})$. Since $\tau(P_i) = \tau(\hat{P}_i) = \tau(R_i)$, this is a contradiction to strategy-proofness. □

Consider the profile $(R_S,\hat{P}_j,P_{N\setminus S\cup j})$. Let $l \in N$ be such that $\tau(P_l) \in (\tau(P_i),s)$ and $\tau(P_{\hat{l}}) \notin (\tau(P_i),\tau(P_l))$ for all $\hat{l} \in N$. By using similar logic as for Claim A.3, we can construct a preference $\hat{P}_l$ such that $\tau(\hat{P}_l) = \tau(P_l)$ and $\varphi_s(R_S,\hat{P}_j,P_{N\setminus S\cup j}) = \varphi_s(R_S,\hat{P}_j,\hat{P}_l,P_{N\setminus S\cup\{j,l\}})$. Continuing in this manner, we can construct a profile $(R_S,\hat{P}_{N\setminus S})$ such that $\tau(\hat{P}_j) = \tau(P_i)$ if $\tau(P_j) \in (\tau(P_i),s)$ and $\hat{P}_j = P_j$ if $\tau(P_j) \notin (\tau(P_i),s)$ and

$$\varphi_s(R_S,\hat{P}_{N\setminus S}) = \varphi_s(R_S,P_{N\setminus S}). \tag{16}$$

Note that by construction of $(R_S,\hat{P}_{N\setminus S})$, there is no agent j such that $\tau(\hat{R}_j) \in (\tau(P_i),s)$. Thus, by applying the same logic as in Case I1, $\varphi_s(R_S,\hat{P}_{N\setminus S}) = \varphi_s(P_1,R_{S\setminus 1},P_{N\setminus S})$. Combining this with (16), we have $\varphi_s(R_S,\hat{P}_{N\setminus S}) = \varphi_s(P_1,R_{S\setminus 1},P_{N\setminus S})$.

■

B. PROOF OF THEOREM 4.1

We make extensive use of two particular types of single-plateaued preferences in our proofs: a single-plateaued preference R is called left (or right) if for all $x < \tau^-(R)$ and all $y > \tau^+(R)$, we have xPy (or yPx).

Lemma B.1. *An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies Pareto optimality if and only if it is unanimous and for all $R_N \in \mathscr{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$, we have $\varphi_{I(R_N)}(R_N) = 1$, where $I(R_N)$ is the minimal interval such that $I(R_N) \cap \tau(R_i) \neq \emptyset$ for all $i \in N$.*

Proof of Lemma B.1. The proof of this lemma is somewhat straightforward. However, for the sake of completeness, we provide it here.

(If part) Suppose φ satisfies Pareto optimality. Then, it is straightforwardly unanimous. Take $R_N \in \mathscr{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$. We show that $\varphi_{I(R_N)}(R_N) = 1$. It is sufficient to show $\varphi_x(R_N) = 0$ for all $x \notin I(R_N)$. Take $x \notin I(R_N)$. Let $I(R_N) = [y,z]$. Assume without loss of generality $x < y$. Since $I(R_N) = [y,z]$, there exists $i \in N$ such that $\tau^+(R_i) = y$. Since R_N is not unanimous, there exists $j \in N$ such that $\tau^-(R_j) > y$. This means yP_jx. Moreover, since $I(R_N) = [y,z]$, $\tau^+(R_i) \geq y$ for all $i \in N$. This means yR_ix for all $i \in N$. Combining, we have y Pareto dominates x. So, $\varphi_x(R_N) = 0$.

(Only-if part) Suppose an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is unanimous and satisfies the property that for all $R_N \in \mathscr{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$, $\varphi_{I(R_N)}(R_N) = 1$. We show that φ satisfies Pareto optimality. Take $R_N \in \mathscr{D}^n$. If R_N is a unanimous profile, then there is nothing to show. Suppose R_N is not unanimous. Take $x \in I(R_N)$. We show that there does not exist $y \in A$ such that yR_ix for all $i \in N$ and yP_jx for some $j \in N$. Assume for contradiction that there is such an alternative $y \in A$. Because $x \in I(R_N)$, there exist $i, j \in N$ such that $\tau^+(R_i) \leq x$ and $\tau^-(R_j) \geq x$. So, if $y > x$, then xP_iy. On the other hand, if $y < x$, then xP_jy. So, y cannot Pareto dominate x.

■

Proof. "Only if" part of the theorem is straightforward, we proceed to prove the "if" part. Let $\mathscr{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. We show that φ is Pareto optimal. Take $R_N \in \mathscr{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$. In view of Lemma B.1, it is sufficient to show that $\varphi_{I(R_N)}(R_N) = 1$. Suppose not. Let $I(R_N) = [x,y]$. Assume without loss of generality, $\varphi_{[1,x-1]}(R_N) > 0$. Take $i \in N$ such that $\tau^-(R_i) > x$. There must exist such an agent since R_N is not a unanimous profile. Let $R'_i \in \mathscr{D}$ be a left-single-plateaued preference such that $\tau^-(R'_i) = x$ and $|\tau(R'_i)| = \kappa_1$.

Claim. $\varphi_{[1,x-1]}(R'_i, R_{N\setminus i}) \geq \varphi_{[1,x-1]}(R_N)$.

Proof of the claim. Note that since R'_i is a left-single-plateaued preference, for all $w \geq \tau^+(R'_i)$ the interval $[1,w]$ is an upper contour set in R'_i. So, by strategy-proofness, $\varphi_{[1,w]}(R'_i, R_{N\setminus i}) \geq \varphi_{[1,w]}(R_N)$, as otherwise agent i manipulates at $(R'_i, R_{N\setminus i})$ via R_i. Now, assume for contradiction that $\varphi_{[1,x-1]}(R'_i, R_{N\setminus i}) < \varphi_{[1,x-1]}(R_N)$. This implies for all $w \geq \tau(R'_i)$,

$$\varphi_{[x,w]}(R'_i, R_{N\setminus i}) > \varphi_{[x,w]}(R_N). \tag{17}$$

Consider the upper contour set $U(x, R_i)$. Since R_i is single-plateaued, there must be an alternative $z \geq \tau^+(R_i)$ such that $U(x, R_i) = [x, \tau^+(R_i)) \cup [\tau^+(R_i)+1, z]$. By 17, we have $\varphi_{[x,z]}(R'_i, R_{N\setminus i}) > \varphi_{[x,z]}(R_N)$. However, since $[x,z]$ is an upper contour set at R_i, this means agent i manipulates at R_N via R'_i. □

Continuing in this manner, we can construct a profile R'_N with $\varphi_{[1,x-1]}(R'_N) > 0$ where $\tau^-(R'_i) = x$ and R'_i is left-single-plateaued for all $i \in \mathscr{D}_i$ with $\tau^-(R_i) > x$ and $R'_i = R_i$ for all other agents. Clearly, $x-1 \notin \cap_{i\in N}\tau(R'_i)$. Moreover, since $I(R_N) = [x,y]$, there must be $i \in N$ with $\tau^+(R_i) = x$. By the construction of R'_N, $R'_i = R_i$ for such an agent i. This means $x+1 \notin \tau(R'_i)$ for such an agent, and consequently, $x+1 \notin \cap_{i\in N}\tau(R'_i)$. Thus, we have $\cap_{i\in N}\tau(R'_i) = \{x\}$. By unanimity, $\varphi_x(R'_N) = 1$, which is a contradiction to $\varphi_{[1,x-1]}(R'_N) > 0$. ■

C. Proof of Theorem 4.2

Proof. Let $\mathscr{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. We show φ is almost plateau-only. Take $i \in N$ and $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in \mathscr{D}^n$ such that $\tau(R_i) = \tau(R'_i)$. Let $x \in A \setminus \tau(R_i)$ be arbitrary. It is enough to show $\varphi_x(R_i, R_{N\setminus i}) = \varphi_x(R'_i, R_{N\setminus i})$. Without loss of generality, assume $x > \tau^+(R_i)$, $\varphi_x(R_i, R_{N\setminus i}) < \varphi_x(R'_i, R_{N\setminus i})$, and $\varphi_y(R_i, R_{N\setminus i}) = \varphi_y(R'_i, R_{N\setminus i})$ for all $y \in (\tau^+(R_i), x)$.

Let $j \in N$ be such that $\tau^+(R_j) \leq \tau^+(R_k)$ for all $k \in N$. Consider $R'_j \in \mathscr{D}$ such that $\tau^+(R'_j) = \tau^+(R_i)$ and R'_j is left single-plateaued.

Claim C.1. *$\varphi_y(R_i, R'_j, R_{N\setminus\{i,j\}}) = \varphi_y(R_i, R_j, R_{N\setminus\{i,j\}})$ and $\varphi_y(R'_i, R'_j, R_{N\setminus\{i,j\}}) = \varphi_y(R'_i, R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$.*

Proof. It is sufficient to show $\varphi_y(R_i, R'_j, R_{N\setminus\{i,j\}}) = \varphi_y(R_i, R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$. The proof of $\varphi_y(R'_i, R'_j, R_{N\setminus\{i,j\}}) = \varphi_y(R'_i, R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$ follows from similar argument. Take $y \in A$

such that $y > \tau^+(R_i)$. Because $\tau^+(R'_j) = \tau^+(R_i)$ and $\tau^+(R_i) < y$, we have $\tau^+(R'_j) < y$. Since R'_j is left single-plateaued and $\tau^+(R'_j) < y$, $U(y,R'_j) = U(y,R_j) \cup [1,\tau^-(R_j))$. By strategy-proofness of φ, we have

$$\varphi_{U(y,R_j)}(R_i,R_j,R_{N\setminus\{i,j\}}) \geq \varphi_{U(y,R_j)}(R_i,R'_j,R_{N\setminus\{i,j\}}) \tag{18}$$

and

$$\varphi_{U(y,R'_j)}(R_i,R'_j,R_{N\setminus\{i,j\}}) \geq \varphi_{U(y,R'_j)}(R_i,R_j,R_{N\setminus\{i,j\}}). \tag{19}$$

As φ is unanimous and strategy-proof, by Theorem 4.1 it follows that φ satisfies Pareto optimality. By Pareto optimality and our assumption on R_j, $\varphi_{[1,\tau^-(R_j))}(R_i,R_j,R_{N\setminus\{i,j\}}) = \varphi_{[1,\tau^-(R_j))}(R_i,R'_j,R_{N\setminus\{i,j\}})$ $= 0$. This together with (18) implies

$$\varphi_{U(y,R'_j)}(R_i,R'_j,R_{N\setminus\{i,j\}}) \leq \varphi_{U(y,R'_j)}(R_i,R_j,R_{N\setminus\{i,j\}}) \tag{20}$$

Now, (19) and (20) imply

$$\varphi_{U(y,R'_j)}(R_i,R'_j,R_{N\setminus\{i,j\}}) = \varphi_{U(y,R'_j)}(R_i,R_j,R_{N\setminus\{i,j\}}). \tag{21}$$

Because $y > \tau^+(R'_j)$, using similar argument for $y-1$, we have

$$\varphi_{U(y-1,R'_j)}(R_i,R_j,R_{N\setminus\{i,j\}}) = \varphi_{U(y-1,R'_j)}(R_i,R'_j,R_{N\setminus\{i,j\}}). \tag{22}$$

Subtracting (22) from (21), we have $\varphi_y(R_i,R_j,R_{N\setminus\{i,j\}}) = \varphi_y(R_i,R'_j,R_{N\setminus\{i,j\}})$. This completes the proof of the claim. ■

Now we complete the proof of the theorem. Let $k \in N$ be such that $\tau^+(R_k) \leq \tau^+(R_l)$ for all $l \in N \setminus \{j\}$. Let $R'_k \in \mathscr{D}$ be such that $\tau^+(R'_k) = \tau^+(R_i)$ and R'_k is left single-plateaued. Using similar logic as for the proof of Claim C.1, we have $\varphi_y(R_i,R'_j,R_k,R_{N\setminus\{i,j,k\}}) = \varphi_y(R_i,R'_j,R'_k,R_{N\setminus\{i,j,k\}})$ and $\varphi_y(R_i,R'_j,R_k,R_{N\setminus\{i,j,k\}}) = \varphi_y(R_i,R'_j,R'_k,R_{N\setminus\{i,j,k\}})$ for all $y > \tau^+(R_i)$.

Continuing in this manner, we construct profiles $(R_i,R'_{N\setminus i})$ and R'_N such that for all $l \in N \setminus i$, $\tau^+(R'_l) = \tau^+(R_i)$ if $\tau^+(R_l) < \tau^+(R_i)$ and $R'_l = R_l$ otherwise and $\varphi_y(R_i,R'_{N\setminus i}) = \varphi_y(R_i,R_{N\setminus i})$ and $\varphi_y(R'_i,R'_{N\setminus i}) = \varphi_y(R'_i,R_{N\setminus i})$ for all $y > \tau^+(R_i)$.

By Pareto optimality, $\varphi_z(R_i,R'_{N\setminus i}) = \varphi_z(R'_i,R'_{N\setminus i}) = 0$ for all $z < \tau^+(R_i)$. Also, by strategy-proofness of φ, $\varphi_{\tau(R_i)}(R_i,R'_{N\setminus i}) = \varphi_{\tau(R_i)}(R'_i,R'_{N\setminus i})$. Since $\varphi_x(R_i,R_{N\setminus i}) < \varphi_x(R'_i,R_{N\setminus i})$, by Claim C.1 we have

$\varphi_{U(x,R_i)}(R_i, R'_{N\setminus i}) < \varphi_{U(x,R_i)}(R'_i, R'_{N\setminus i})$. This means i manipulates at $(R_i, R'_{N\setminus i})$ via R'_i, which contradicts that φ is strategy-proof. ■

D. Proof of Theorem 4.3

Proof. (If part) Let $\mathscr{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be an RSCF satisfying generalized uncompromisingness. We show that φ is strategy-proof. Take $R_N \in \mathscr{D}^n$ and $R'_i \in \mathscr{D}$. By Remark 4.3, φ is almost plateau-only. Thus, if $\tau(R_i) = \tau(R'_i)$, $\varphi_{U(x,R_i)}(R_i, R_{N\setminus i}) = \varphi_{U(x,R_i)}(R'_i, R_{N\setminus i})$ for all $x \in A$. We distinguish the following cases based on the relative position of $\tau(R_i)$ and $\tau(R'_i)$ to complete the proof.

Case 1. Suppose $\tau^-(R_i) < \tau^-(R'_i)$ and $\tau^+(R_i) < \tau^+(R'_i)$

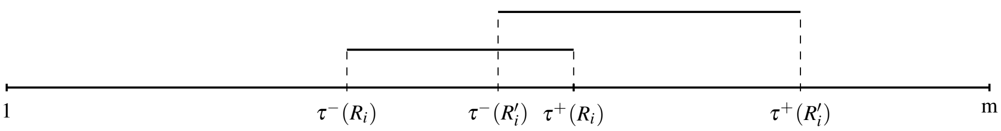

Figure 2: $\tau^-(R_i) < \tau^-(R'_i)$ and $\tau^+(R_i) < \tau^+(R'_i)$.

By Remark 4.1,

$$\varphi_x(R_i, R_{N\setminus i}) = \varphi_x(R'_i, R_{N\setminus i}) \text{ for all } x \notin [\tau^-(R_i), \tau^+(R'_i)]. \tag{23}$$

Because S is single-plateaued , for all $y \in A$, there are $y' \leq \tau^-(R_i)$ and $y'' \geq \tau^+(R_i)$ such that $U(y, R_i) = [y', \tau^-(R_i) - 1] \cup [\tau^-(R_i), y'']$, where $[y', \tau^-(R_i) - 1] = \emptyset$ when $y' = \tau^-(R_i)$. This together with (23), implies that to show φ is strategy-proof it is sufficient to show $\varphi_{[\tau^-(R_i),y]}(R_i, R_{N\setminus i}) \geq \varphi_{[\tau^-(R_i),y]}(R'_i, R_{N\setminus i})$ for all $y \in [\tau^+(R_i), \tau^+(R'_i)]$.

Take $y = \tau^+(R_i)$. It follows from (23) that $\varphi_{[\tau^-(R_i),\tau^+(R'_i)]}(R_i, R_{N\setminus i}) = \varphi_{[\tau^-(R_i),\tau^+(R'_i)]}(R'_i, R_{N\setminus i})$. Now, take $y \in [\tau^+(R_i), \tau^+(R'_i) - 1]$. As $\tau^+(R_i) < y + 1$ and $\tau^+(R'_i) \geq y + 1$, by condition (ii) of Definition 4.6, we have

$$\varphi_{[y+1,m]}(R_i, R_{N\setminus i}) \leq \varphi_{[y+1,m]}(R'_i, R_{N\setminus i}). \tag{24}$$

By condition (i) of Definition 4.6, $\varphi_{[\tau^-(R_i),m]}(R_i, R_{N\setminus i}) = \varphi_{[\tau^-(R_i),m]}(R'_i, R_{N\setminus i})$. This together with (24), implies

$$\varphi_{[\tau^-(R_i),y]}(R_i, R_{N\setminus i}) \geq \varphi_{[\tau^-(R_i),y]}(R'_i, R_{N\setminus i}).$$

Case 2. Suppose $\tau^-(R_i) > \tau^-(R'_i)$ and $\tau^+(R_i) > \tau^+(R'_i)$.

This case is very similar to Case 1, however for the sake of completeness we provide a formal proof. Take

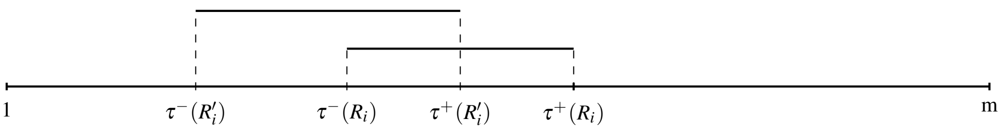

Figure 3: $\tau^-(R_i) > \tau^-(R_i')$ and $\tau^+(R_i) > \tau^+(R_i')$.

$x \notin [\tau^-(R_i')+1, \tau^+(R_i)]$. By Condition (i) of Definition 4.6, we have

$$\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \varphi_{[x,m]}(R_i', R_{N\setminus i}). \tag{25}$$

Using similar logic as in Case 1, to show φ is strategy-proof, it is sufficient to show $\varphi_{[y,\tau^+(R_i)]}(R_i, R_{N\setminus i}) \geq \varphi_{[y,\tau^+(R_i)]}(R_i', R_{N\setminus i})$ for all $y \in [\tau^-(R_i'), \tau^-(R_i)]$. First, take $y = \tau^-(R_i')$. By Condition (i) of Definition 4.6, it follows that $\varphi_{[\tau^-(R_i'),\tau^+(R_i)]}(R_i, R_{N\setminus i}) = \varphi_{[\tau^-(R_i'),\tau^+(R_i)]}(R_i', R_{N\setminus i})$. Next, take $y \in [\tau^-(R_i')+1, \tau^-(R_i)]$. As $\tau^-(R_i) \geq y$ and $\tau^-(R_i') < y$, by Condition (ii) of Definition 4.6, we have

$$\varphi_{[y,m]}(R_i, R_{N\setminus i}) \geq \varphi_{[y,m]}(R_i', R_{N\setminus i}). \tag{26}$$

Taking $x = \tau^+(R_i)+1$ in (25) and subtracting it from (26), we get

$$\varphi_{[y,\tau^+(R_i)]}(R_i, R_{N\setminus i}) \geq \varphi_{[y,\tau^+(R_i)]}(R_i', R_{N\setminus i}).$$

Case 3. Suppose $\tau^-(R_i) \geq \tau^-(R_i')$ and $\tau^+(R_i) \leq \tau^+(R_i')$.
The proof for this case follows by combining the arguments in Case 1 and Case 2. For the sake of completeness, we provide a formal proof here.

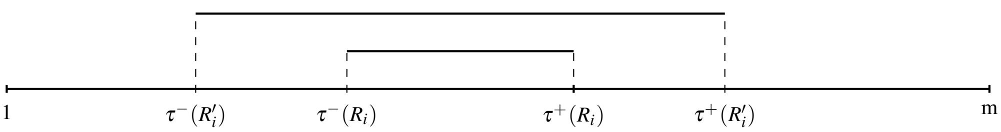

Figure 4: $\tau^-(R_i) \geq \tau^-(R_i')$ and $\tau^+(R_i) \leq \tau^+(R_i')$.

Case 4. Suppose $\tau^-(R_i) \leq \tau^-(R_i')$ and $\tau^+(R_i) \geq \tau^+(R_i')$.

By Remark 4.1, $\varphi_x(R_i, R_{N\setminus i}) = \varphi_x(R_i', R_{N\setminus i})$ for all $x \notin \tau(R_i)$. Therefore, agent i cannot manipulate at $(R_i, R_{N\setminus i})$ via R_i'.

■

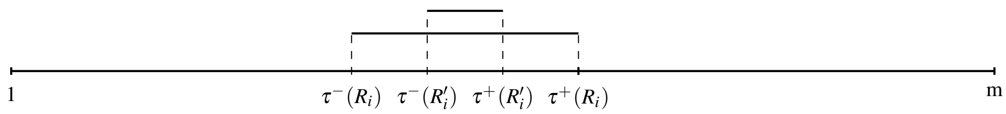

Figure 5: $\tau^-(R_i) \leq \tau^-(R_i')$ and $\tau^+(R_i) \geq \tau^+(R_i')$.

E. Proof of Theorem 4.4

Proof. "If" part of the theorem follows from Theorem 4.3. We proceed to prove the "only if" part. Let $\varphi : \mathscr{D}^n \to \Delta A$ be a strategy-proof and unanimous RSCF. We show φ satisfies generalized uncompromisingness. Take $i \in N$ and $R_i, R_i' \in \mathscr{D}$ with $\tau^+(R_i) < \tau^+(R_i')$, $R_{N\setminus i} \in \mathscr{D}^{n-1}$, and $x \in A$.

We show that $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \varphi_{[x,m]}(R_i', R_{N\setminus i})$ if $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$ or if $\min\{\tau^-(R_i), \tau^-(R_i')\} \geq x$. Suppose $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$. In view of Theorem 4.2, we assume both R_i, R_i' to be left single-plateaued. By strategy-proofness, $\varphi_{U(x-1,R_i)}(R_i, R_{N\setminus i}) \geq \varphi_{U(x-1,R_i)}(R_i', R_{N\setminus i})$ and $\varphi_{U(x-1,R_i')}(R_i', R_{N\setminus i}) \geq \varphi_{U(x-1,R_i')}(R_i, R_{N\setminus i})$. However, since both R_i and R_i' are left single-plateaued, $U(x-1,R_i) = U(x-1,R_i') = [1,x-1]$. Therefore, $\varphi_{[x,m]}(R_i', R_{N\setminus i}) = \varphi_{[x,m]}(R_i, R_{N\setminus i})$.

For the case where $\min\{\tau^-(R_i), \tau^-(R_i')\} \geq x$, by means of Theorem 4.2 we can assume both R_i and R_i' to be right single-plateaued. Then $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \varphi_{[x,m]}(R_i', R_{N\setminus i})$ follows by using similar argument as above.

Now we show (ii) in Definition 4.6. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$. Since $x > \tau^+(R_i)$ in view of Theorem 4.2 , we assume R_i to be left single-plateaued. By strategy-proofness, $\varphi_{U(x-1,R_i)}(R_i, R_{N\setminus i}) \geq \varphi_{U(x-1,R_i)}(R_i', R_{N\setminus i})$. However, as R_i is a left single-plateaued preference, $U(x-1,R_i) = [1,x-1]$. This means $\varphi_{[x,m]}(R_i, R_{N\setminus i}) \leq \varphi_{[x,m]}(R_i', R_{N\setminus i})$.

Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$. Since $x \leq \tau^-(R_i')$ in view of Theorem 4.2, without loss of generality we assume R_i' to be right single-plateaued. Then, $U(x-1,R_i) = [x,m]$. By strategy-proofness, this means $\varphi_{[x,m]}(R_i', R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i, R_{N\setminus i})$ which completes the proof. ■

F. Proof of Theorem 4.5

Proof. (If part) In Section 4.5, we introduce a generalization of random plateau rules which we call generalized k-random plateau rules. We show those rules to be unanimous and strategy-proof. If part of the theorem follows from that.

(Only-if part) Suppose an RSCF $\varphi : \mathscr{D}^2 \to \Delta A$ is unanimous and strategy-proof. We show that φ is a random plateau rule. Condition (i) of Definition 4.7 follows from unanimity of φ.

Consider a preference profile $R_N \in \mathscr{D}^2$ with $\tau^+(R_1) < \tau^-(R_2)$. It follows from Theorem 4.1 that φ is Pareto optimal. By means of Pareto optimality, we have $\varphi_x(R_N) = 0$ for all $x \notin [\tau^+(R_1), \tau^-(R_2)]$. We prcoceed to show $\varphi_x(R_N) = \beta_2[x,m]$ for some $\beta_2 \in \Delta A$ where $x \in A$ such that $\tau^+(R_1) < x \leq \tau^-(R_2)$. Let $\bar{R}_N \in \mathscr{D}^2$ be such that $\tau(\bar{R}_1) = [1, \kappa_1]$ and $\tau(\bar{R}_2) = [m - \kappa_1 + 1, m]$. Define $\varphi(\bar{R}_N) := \beta_2$. Consider the profile $(R_1, \bar{R}_2)$. Since $\tau^+(R_1) < x$, by strategy-proofness, $\varphi_{[x,m]}(R_1, \bar{R}_2) = \beta_2[x,m]$. Again, since $x < \tau^-(R_2)$, $\varphi_{[x,m]}(R_1, R_2) = \beta_2[x,m]$. This proves Condition (ii) of Definition 4.7. Condition (iii) of Definition 4.7 can be proved with a similar argument. ■

G. Proof of Theorem 4.6

Proof. Let $k \in \{0, \ldots, \kappa_1 - 1\}$ and let $(\beta_S)_{S \subseteq N}$ be a collection of monotonic parameters. Suppose $\varphi : \mathscr{D}^n \to \Delta A$ is a k-random plateau rule with respect to $(\beta_S)_{S \subseteq N}$. We show that φ is unanimous and strategy-proof. Unanimity of φ follows from Condition (i) of Definition 4.9. To show that φ is strategy-proof, by Theorem 4.3, it is enough to show that it satisfies generalized uncompromisingness. Consider $R_i, R'_i \in \mathscr{D}_i$ and $R_{N \setminus \{i\}} \in \mathscr{D}^{n-1}$. Suppose $x \in A$ is such that $x \leq \min\{\tau^-(R_i), \tau^-(R'_i)\}$. Let $S \subseteq N$ be such that $i \in S$ if and only if $\tau^-(R_i) \geq x - k$ and $\tau^+(R_i) \geq x$ and let $S' \subseteq N$ be such that $i \in S'$ if and only if $\tau^-(R'_i) \geq x - k$ and $\tau^+(R'_i) \geq x$. Since, $x \leq \min\{\tau^-(R_i), \tau^-(R'_i)\}$, we have $S = S'$. By the definition of φ, $\varphi_{[x,m]}(R_i, R_{N \setminus \{i\}}) = \beta_S[x,m]$. One can similarly show that $\varphi_{[x,m]}(R_i, R_{N \setminus \{i\}}) = \varphi_{[x,m]}(R'_i, R_{N \setminus \{i\}})$ where $x \in A$ is such that $x > \max\{\tau^+(R_i), \tau^+(R'_i)\}$. Now consider $x \in A$ such that $\tau^+(R_i) < x \leq \tau^+(R'_i)$. Let $S \subseteq N$ be such that $i \in S$ if and only if $\tau^-(R_i) \geq x - k$ and $\tau^+(R_i) \geq x$ and let $S' \subseteq N$ be such that $i \in S'$ if and only if $\tau^-(R'_i) \geq x - k$ and $\tau^+(R'_i) \geq x$. Because $\tau^+(R_i) < x \leq \tau^+(R'_i)$, $S \subseteq S'$. By the definition of φ, $\varphi_{[x,m]}(R_i, R_{N \setminus \{i\}}) = \beta_S[x,m]$ and $\varphi_{[x,m]}(R'_i, R_{N \setminus \{i\}}) = \beta_{S'}[x,m]$. By the definition of $(\beta_S)_{S \subseteq N}$, $\beta_S[x,m] \leq \beta_{S'}[x,m]$, which means $\varphi_{[x,m]}(R_i, R_{N \setminus \{i\}}) \leq \varphi_{[x,m]}(R'_i, R_{N \setminus \{i\}})$. Similarly, $\varphi_{[x,m]}(R_i, R_{N \setminus \{i\}}) \leq \varphi_{[x,m]}(R'_i, R_{N \setminus \{i\}})$ where $x \in A$ is such that $\tau^-(R_i) < x \leq \tau^-(R'_i)$. ■

H. Proof of Theorem 4.7

Proof. (If part) Let $\mathscr{D}$ be a κ-single-plateaued domain for some $\kappa \in \{1, \ldots, m\}$ and let $\varphi : \mathscr{D}^n \to \Delta A$ be a κ-plateaued rule . By definition φ is anonymous and plateau-only. To show that φ is strategy-proof, in view of Theorem 4.3, it is enough to show that it satisfies generalized uncompromisingness. Consider $i \in N$, $R_i, R'_i \in \mathscr{D}$ with $\tau^+(R_i) < \tau^+(R'_i)$, $R_{N \setminus i} \in \mathscr{D}^{n-1}$, and $x \in A$. First we show that Condition (i) in Definition 4.6 holds. Consider the case where $\tau^+(R'_i) < x$ or $\tau^-(R_i) \geq x$. As $\tau^+(R_i) < \tau^+(R'_i)$, it follows that when $\tau^+(R_i') < x$ we have $\tau^+(R_i) < x$, and when $\tau^-(R_i) \geq x$ we have $\tau^-(R_i') > x$. This implies $n_l^x(R_i, R_{N\setminus i}) = n_l^x(R_i', R_{N\setminus i})$ for all $0 \leq l \leq \kappa - 1$. By the definition of κ-plateaued rule , this yields $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \varphi_{[x,m]}(R_i', R_{N\setminus i})$, which proves that φ satisfies Condition (i) in Definition 4.6.

Next we show φ satisfies Condition (ii) in Definition 4.6. We distinguish the following two cases.

Case 1. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$.

By the definition of κ-plateaued rule, $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\varphi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. Since $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$, there must exist $0 \leq l' \leq \kappa - 1$ such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq l'$, and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $l' < l \leq \kappa - 1$. This implies $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$. Therefore, by Condition (ii) of Definition 4.10, we have $\beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n} \oplus 1, x)$, and hence it follows that $\varphi_{[x,m]}(R_i', R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i, R_{N\setminus i})$.

Case 2. Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$.

By the definition of κ-plateaued rule, $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\varphi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. If $\tau^+(R_i) < x$, then $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq \kappa - 1$. However, if $x \leq \tau^+(R_i) < x + \kappa - 1$, then there must exist l' such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $0 \leq l \leq l'$ and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $l' < l \leq \kappa - 1$. In both these cases, $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$, and hence by using Condition (ii) of Definition 4.10, we have $\varphi_{[x,m]}(R_i', R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i, R_{Nx\setminus i})$.

(Only-if part) Let φ be a strategy-proof, anonymous, and plateau-only RSCF on $\mathscr{D}^n$. We show that it is a κ-plateaued rule.

Lemma H.1. *Let $x \in A$ and $R_N, R_N' \in \mathscr{D}^n$ be such that $n_l^x(R_N) = n_l^x(R_N')$ for all $0 \leq l \leq \kappa - 1$. Then, $\varphi_{[x,m]}(R_N) = \varphi_{[x,m]}(R_N')$.*

Proof. Note that for all $\bar{R}_N \in \{R_N, R_N'\}$, $|\{i \in N \mid \tau^+(\bar{R}_i) = x + l\}| = n_l^x(\bar{R}_N) - n_{l+1}^x(\bar{R}_N)$ for all $0 \leq l \leq \kappa - 2$ and $|\{i \in N \mid \tau^+(\bar{R}_i) < x\}| = n - n_0^x(\bar{R}_N)$. Because $n_l^x(R_N) = n_l^x(R_N')$ for all $0 \leq l \leq \kappa - 1$, this means $|\{i \in N \mid \tau^+(R_i) = x + l\}| = |\{i \in N \mid \tau^+(R_i') = x + l\}|$ for all $0 \leq l \leq \kappa - 2$, $|\{i \in N \mid \tau^-(R_i) \geq x\}| = |\{i \in N \mid \tau^-(R_i') \geq x\}|$ and $|\{i \in N \mid \tau^+(R_i) < x\}| = |\{i \in N \mid R_i'(1) < x\}|$. Let $n_l = |\{i \in N \mid \tau^+(R_i) = x + l\}|$ for all $0 \leq l \leq \kappa - 2$. Since φ is anonymous, assume without loss of generality that

(i) $\{i \in N \mid \tau^+(R_i) = x + l\} = \{i \in N \mid \tau^+(R_i') = x + l\} = \{i_{s_{l-1}} + 1, \ldots, i_{s_l}\}$, where for all $0 \leq l \leq \kappa - 2$, $s_l = n_0 + \ldots + n_l$ and $i_{s_{-1}} = 0$, and

(ii) for all $i \in N \setminus \{i_{s_{\kappa-2}}, \ldots, 1\}$, either $\tau^+(R_i), \tau^+(R_i') < x$ or $\tau^-(R_i), \tau^-(R_i') \geq x$. In view of this, it is enough to show that $\varphi_{[x,m]}(R_i, R_{N\setminus i}) = \varphi_{[x,m]}(R_i', R_{N\setminus i})$ for all $(R_i, R_{N\setminus i}), (R_i', R_{N\setminus i}) \in S$ such that either $\tau^+(R_i), \tau^+(R_i') < x$ or $\tau^-(R_i), \tau^-(R_i) \geq x$. Suppose $\tau^+(R_i), \tau^+(R_i') < x$. Since φ is plateau-only, assume without loss of generality that both R_i and R_i' are left single-plateaued. This means $U(x-1, R_i) =$

$U(x-1,R_i') = [1,x-1]$. Now, by straightforward application of strategy-proofness, it follows that $\varphi_{[1,x-1]}(R_i,R_{N\setminus i}) = \varphi_{[1,x-1]}(R_i',R_{N\setminus i})$. Thus, $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \varphi_{[x,m]}(R_i',R_{N\setminus i})$.

Next, suppose $\tau^-(R_i), \tau^-(R_i) \geq x$. Since φ is plateau-only, assume without loss of generality that both R_i and R_i' are right single-plateaued. This means $U(x,R_i) = U(x,R_i') = [x,m]$. Now, by straightforward application of strategy-proofness, it follows that $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \varphi_{[x,m]}(R_i',R_{N\setminus i})$. This completes the proof of the lemma. ■

In view of Lemma H.1, for all $(\underset{\sim}{n},x) \in \mathscr{F}(\kappa)$, define $\beta(\underset{\sim}{n},x) = \varphi_{[x,m]}(R_N)$ where R_N is such that $n_l^x(R_i,R_{N\setminus i}) = \underset{\sim}{n}_l$ for all $0 \leq l \leq \kappa - 1$. In what follows we show that the parameters βs are plateau parameters.

We show that $(\beta(\underset{\sim}{n},x))$ satisfies Condition (ii) in Definition 4.10. In view of Lemma H.1, it is enough to show that $\varphi_{[x,m]}(R_i,R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i',R_{N\setminus i})$ where $(R_i,R_{N\setminus i}),(R_i',R_{N\setminus i}) \in \mathscr{D}^n$ are such that either $\left[\tau^+(R_i) < x \text{ and } \tau^+(R_i') \geq x\right]$ or $\left[\tau^-(R_i) \geq x \text{ and } x \leq \tau^+(R_i') < x+\kappa-1\right]$. First we consider the case where $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$. In view of Lemma H.1 , it is enough to show that $\varphi_{[x,m]}(R_i,R_{N\setminus i}) \leq \varphi_{[x,m]}(R_i',R_{N\setminus i})$ where $(R_i,R_{N\setminus i}),(R_i',R_{N\setminus i}) \in S$ are such that $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$. Since φ is plateau-only, assume without loss of generality that R_i is left single-plateaued. Then $U(x-1,R_i) = [1,x-1]$. By strategy-proofness, $\varphi_{U(x-1,R_i)}(R_i,R_{N\setminus i}) \geq \varphi_{U(x-1,R_i)}(R_i',R_{N\setminus i})$, which means $\varphi_{[x,m]}(R_i,R_{N\setminus i}) \leq \varphi_{[x,m]}(R_i',R_{N\setminus i})$. Next we consider the case where $\tau^-(R_i) \geq x$ and $x \leq \tau^+(R_i') < x+\kappa-1$. Next we show that $(\beta(\underset{\sim}{n},x))$ satisfies Condition (ii) in Definition 4.10. In view of Lemma H.1, it is enough to show that $\varphi_{[x,m]}(R_i,R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i',R_{N\setminus i})$ where $(R_i,R_{N\setminus i}),(R_i',R_{N\setminus i}) \in \mathscr{D}^n$ are such that $\tau^-(R_i) \geq x$ and $x \leq \tau^+(R_i') < x+\kappa-1$. Since φ is plateau-only, assume without loss of generality that R_i is right single-plateaued. Then $U(x,R_i) = [x,m]$. By strategy-proofness, $\varphi_{U(x,R_i)}(R_i,R_{N\setminus i}) \geq \varphi_{U(x,R_i)}(R_i',R_{N\setminus i})$, which means $\varphi_{[x,m]}(R_i,R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i',R_{N\setminus i})$.

Finally, we show that $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ satisfies Condition (i) of Definition 4.10. Take $\underset{\sim}{n},\underset{\sim}{n}^+ \in \underset{\sim}{N}$, $R_N \in \mathscr{D}^n$ and $x \in [3,m]$ such that $n_l^x(R_N) = \underset{\sim}{n}_l$ and $\underset{\sim}{n}_l^{x-1}(R_N) = \underset{\sim}{n}_l^+$ for all $0 \leq l \leq \kappa-1$. It is easy to see that such a R_N and x exist for every possible choice of $\underset{\sim}{n}$ and $\underset{\sim}{n}^+$. Since φ is an RSCF, $\varphi_{[x,m]}(R_N) \leq \varphi_{[x-1,m]}(R_N)$. By the construction of $(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}$, we have $0 \leq \beta(\underset{\sim}{n},x) \leq \beta(\underset{\sim}{n}^+,x-1) \leq 1$ for all $\underset{\sim}{n},\underset{\sim}{n}^+ \in \underset{\sim}{N}$. ■

I. PROOF OF COROLLARY 4.1

Proof. It remains to show that a κ-plateaued rule is unanimous if and only if it is based on a collection of unanimous plateau parameters.

(If part) Suppose $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ is unanimous and φ is a κ-plateaued rule with respect to $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$. We show φ is unanimous.

Let $R_N \in \mathscr{D}^n$ be such that $\cap_{i\in N}\tau(R_i) \neq \emptyset$. Let $\cap_{i\in N}\tau(R_i) = [y,z]$. Then, $n_0^y(R_N) = n$ and $n_{\kappa-1}^y(R_N) > 0$ (see Figure 6 for details). Since $\underset{\sim}{\hat{n}}_0 = n$ and $\underset{\sim}{\hat{n}}_{\kappa-1} > 0$, by unanimity of plateau parameters, we have $\beta(\underset{\sim}{\hat{n}},y) = 1$. Because $\varphi_{[y,m]}(R_N) = \beta(\underset{\sim}{\hat{n}},y)$, this means $\varphi_{[y,m]}(R_N) = 1$. As $\cap\tau(R_i) = [y,z]$, $n_0^{z+1}(R_N) < n$ and $n_{\kappa-1}^{z+1}(R_N) = 0$ (see Figure 6 for details). Since $\underset{\sim}{\bar{n}}_0 < n$ and $\underset{\sim}{\bar{n}}_{\kappa-1} = 0$, by unanimity of plateau parameters, we have $\beta(\underset{\sim}{\bar{n}},z+1) = 0$. Because $\varphi_{[z+1,m]}(R_N) = \beta(\underset{\sim}{\bar{n}},z+1)$, this means $\varphi_{[z+1,m]}(R_N) = 0$. So, we have $\varphi_{[y,z]}(R_N) = 1$.

(Only-if part) Let φ be a strategy-proof, plateau-only, anonymous and unanimous RSCF on $\mathscr{D}^n$. We show it is a κ-plateaued rule with respect to unanimous plateau parameters. By Theorem 4.8, φ is a κ-plateaued rule. Let $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ be the plateau parameters of φ. We need to show that the collection $\{(\beta(\underset{\sim}{n},x))_{(\underset{\sim}{n},x)\in\mathscr{F}(\kappa)}\}$ is unanimous.

Since, φ is unanimous and strategy-proof, by Theorem 4.1, it is Pareto optimal. Take $R_N \in \mathscr{D}^n$ and $x \in [2,m]$ such that $n_0^x(R_N) = n$ and $n_{\kappa-1}^x(R_N) > 0$. Since, $n_0^x(R_N) = n$, $\min_{i\in N}(\tau^+(R_i)) \geq x$. Also, since $n_{\kappa-1}^x(R_N) > 0$, there must exist $\hat{i} \in N$ such that $\tau^-(R_{\hat{i}}) \geq x$. Take $y < x$. As $\min_{i\in N}(\tau^+(R_i)) \geq x$, xR_iy for all $i \in N$. Moreover, because $\tau^-(R_{\hat{i}}) \geq x$, $xP_{\hat{i}}y$. By Pareto optimality of φ, $\varphi_y(R_N) = 0$. Since, $y < x$ is arbitrary, this means $\varphi_{[x,m]}(R_N) = 1$. Because $\varphi_{[x,m]}(R_N) = \beta(\underset{\sim}{n},x)$ where $(\underset{\sim}{n},x)$ is such that $\underset{\sim}{n}_0 = n$ and $\underset{\sim}{n}_{\kappa-1} > 0$, it follows that $\beta(\underset{\sim}{n},x) = 1$.

Now take $R_N \in \mathscr{D}^n$ such that $n_0^x < n$ and $n_{\kappa-1}^x = 0$. Since $n_0^x < n$, this means there exists $\bar{i} \in N$ such that $\tau^+(R_{\bar{i}}) < x$. Moreover, as $n_{\kappa-1}^x = 0$, $\min_{i\in N}\tau^-(R_i) < x$. Take $y \geq x$. As $\min_{i\in N}\tau^-(R_i) < x$, $\min_{i\in N}\tau^-(R_i)R_iy$ for all $i \in N$. Also, as $\tau^+(R_{\bar{i}}) < x$, this means $\tau^+(R_{\bar{i}})P_{\bar{i}}y$. By Pareto optimality, $\varphi_y(R_N) = 0$. Since $y \geq x$ is arbitrary, we have $\varphi_{[x,m]}(R_N) = 0$. Because $\varphi_{[x,m]}(R_N) = \beta(\underset{\sim}{n},x)$ where $(\underset{\sim}{n},x)$ is such that $\underset{\sim}{n}_0 < n$ and $\underset{\sim}{n}_{\kappa-1} = 0$, it follows that $\beta(\underset{\sim}{n},x) = 0$.

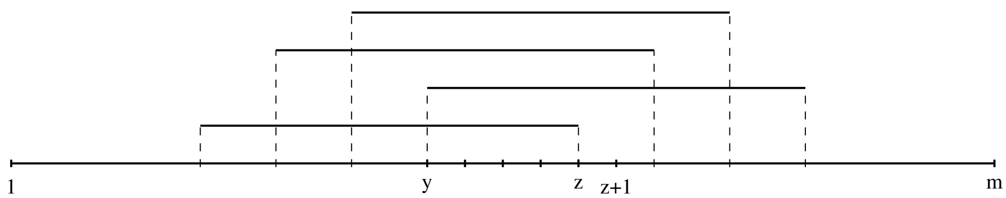

Figure 6

■

J. PROOF OF THEOREM 4.8

Proof. Let $\mathscr{D}$ be a $(\kappa,\hat{\kappa})$-single-plateaued domain for some $\kappa \in \{1,\ldots,m\}$ and some $\hat{\kappa} \geq \kappa$ and let $\varphi : \mathscr{D}^n \to \Delta A$ be a κ-plateaued rule. By definition φ is anonymous and plateau-only. To show that φ is strategy-proof, by Theorem 4.3, it is enough to show that it satisfies generalized uncompromisingness. Consider $i \in N$, $R_i, R_i' \in \mathscr{D}$, $R_{N\setminus i} \in \mathscr{D}^{n-1}$, and $x \in A$. First we show that Condition (i) in Definition 4.6 holds. We distinguish the following two cases.

Case 1. Suppose $x \leq \min\{\tau^-(R_i), \tau^-(R_i')\}$.

As $\mathscr{D}$ is a $(\kappa,\hat{\kappa})$-single-plateaued domain, $|\tau(R_i)| \geq \kappa$ and $|\tau(R_i')| \geq \kappa$. This implies $n_l^x(R_i,R_{N\setminus i}) = n_l^x(R_i',R_{N\setminus i})$ for all $0 \leq l \leq \kappa-1$. By the definition of κ-plateaued rule, this yields $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \varphi_{[x,m]}(R_i',R_{N\setminus i})$.

Case 2. Suppose $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$.

This implies $n_l^x(R_i,R_{N\setminus i}) = n_l^x(R_i',R_{N\setminus i})$ for all $0 \leq l \leq \kappa-1$. By the definition of κ-plateaued rule, this yields $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \varphi_{[x,m]}(R_i',R_{N\setminus i})$.

Next we show φ satisfies Condition (ii) in Definition 4.6. We distinguish the following two cases.

Case 1. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$.

By the definition of κ-plateaued rule, $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \beta(\underset{\sim}{n},x)$ and $\varphi_{[x,m]}(R_i',R_{N\setminus i}) = \beta(\underset{\sim}{n}',x)$, where for all $0 \leq l \leq \kappa-1$, $\underset{\sim}{n}_l = n_l^x(R_i,R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i',R_{N\setminus i})$. Since $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$, there must exist $0 \leq l' \leq \kappa-1$ such that $n_l^x(R_i',R_{N\setminus i}) = n_l^x(R_i,R_{N\setminus i})+1$ for all $0 \leq l \leq l'$, and $n_l^x(R_i',R_{N\setminus i}) = n_l^x(R_i,R_{N\setminus i})$ for all $l' < l \leq \kappa-1$. This implies $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$. Therefore, by Condition (ii) of Definition 4.10, we have $\beta(\underset{\sim}{n},x) \leq \beta(\underset{\sim}{n}\oplus 1,x)$, and hence it follows that $\varphi_{[x,m]}(R_i',R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i,R_{N\setminus i})$.

Case 2. Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$.

By the definition of κ-plateaued rule, $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \beta(\underset{\sim}{n},x)$ and $\varphi_{[x,m]}(R_i',R_{N\setminus i}) = \beta(\underset{\sim}{n}',x)$, where for all $0 \leq l \leq \kappa-1$, $\underset{\sim}{n}_l = n_l^x(R_i,R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i',R_{N\setminus i})$. If $\tau^+(R_i) < x$, then $n_l^x(R_i',R_{N\setminus i}) = n_l^x(R_i,R_{N\setminus i})+1$ for all $0 \leq l \leq \kappa-1$. However, if $x \leq \tau^+(R_i) < x+\kappa-1$, then there must exist l' such that $n_l^x(R_i',R_{N\setminus i}) = n_l^x(R_i,R_{N\setminus i})$ for all $0 \leq l \leq l'$ and $n_l^x(R_i',R_{N\setminus i}) = n_l^x(R_i,R_{N\setminus i})+1$ for all $l' < l \leq \kappa-1$. In both these cases, $\underset{\sim}{n}' = \underset{\sim}{n}\oplus 1$, and hence by using Condition (ii) of Definition 4.10, we have $\varphi_{[x,m]}(R_i',R_{N\setminus i}) \geq \varphi_{[x,m]}(R_i,R_{Nx\setminus i})$. If $\tau^+(R_i) \geq x+\kappa-1$, $n_l^x(R_i,R_{N\setminus i}) = n_l^x(R_i',R_{N\setminus i})$ for all $0 \leq l \leq \kappa-1$. By the definition of κ-plateaued rule, this yields $\varphi_{[x,m]}(R_i,R_{N\setminus i}) = \varphi_{[x,m]}(R_i',R_{N\setminus i})$.

■

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